**Abstract:** In this paper the notion of \( \hat{\delta} \omega \) closed sets is introduced and some of its basic properties are studied. This new class of sets is independent of semi closed and closed sets. Also the relationship with some of the known closed sets is discussed.

**Keywords:** \( \hat{\delta} \omega \) closed sets, closed sets, \( \omega \) closed sets.

### I. INTRODUCTION

Levine, velicko introduced the notions of generalized closed (briefly gclosed) and \( \delta \) closed sets respectively and studied their basic properties. The notion of \( I_g \) closed sets was first introduced by Dontchev in 1999. Navaneetha Krishnan and Joseph further investigated and characterized \( I_g \) closed sets. Jan D. Dontchev and Maximilian Ganster, Yuksel, Acikgoz and Noiri introduced and studied the notions of \( \delta \) generalized closed (briefly \( \delta g \) closed) and \( \delta -I \)-closed sets respectively. The purpose of this paper is to define a new class of sets called \( \hat{\delta} \omega \) closed sets and also study some basic properties and characterizations. Throughout this paper \((X, \tau, I)\) represents an ideal topological space on which no separation axiom is assumed unless otherwise mentioned. For a subset \( A \) of a ideal topological space \( X \), \( \text{cl}(A) \) and \( \text{int}(A) \) denote the closure of \( A \) and the interior of \( A \) respectively. \( X \setminus A \) or \( A^c \) denotes the complement of \( A \) in \( X \). We recall the following definitions and results.

### II. PRELIMINARIES

**A.** Subset \( A \) of a space \( X \) is called

- pre-open set if \( A \subseteq \text{int}(A) \) and pre-closed set if \( \text{cl}(A) \subseteq A \).
- semi-open set if \( A \subseteq \text{cl}(A) \) and semi-closed set if \( \text{int}(A) \subseteq A \).

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- regular open set if \( A = \text{int}(A) \) and regular closed set if \( A = \text{cl}(A) \).
- \( \Pi \)-open set if \( A \) is a finite union of regular open sets.
- regular semi open if there is a regular open \( U \) such \( U \subseteq A \subseteq \text{cl}(U) \).

**B.** Subset \( A \) of \((X, \tau, I)\) is called

- generalized closed set, if \( \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is open in \( X \).
- regular generalized closed set, if \( \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is regular open in \( X \).
- weakly generalized closed set, if \( \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is open in \( X \).
- weakly closed set, if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi open in \( X \).
- regular weakly generalized closed set, if \( \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is regular open in \( X \).
- regular weakly closed set, if \( \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is regular semi open.
- g-closed if \( \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is w-open.

Let \( A \) and \( B \) be subsets of an ideal topological space \((X, \tau, I)\). Then, the following properties holds.

- \( A \subseteq \sigma \text{cl}(A) \).
- If \( A \subseteq B \), then \( \sigma \text{cl}(A) \subseteq \sigma \text{cl}(B) \).
- \( \sigma \text{cl}(A) = \bigcap \{ F \subseteq X : A \subseteq F \text{ and } F \text{ is } \delta - I \text{-closed} \} \).
- If \( A \) is \( \delta -I \)-closed set of \( X \) for each \( \alpha \in \Delta \), then \( \bigcap \{A\alpha : \alpha \in \Delta \} \) is \( \delta -I \)-closed.
- \( \sigma \text{cl}(A) \) is \( \delta -I \)-closed.
- \( \delta -I \) closure is \( \{ x \in X : \text{int}(\text{cl}^*(U)) \cap A \neq \phi, U \in I \} \).
III. $\tilde{\delta}\omega$ - CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

Definition 3.1: A subset $A$ of an ideal space $(X, \tau, I)$ is called $\tilde{\delta}\omega$ closed, if $\sigma cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\omega$ open.

Theorem 3.1

Every g-closed set in $X$ is $\tilde{\delta}\omega$ -closed set in $X$.

Proof: Let $A$ be an arbitrary g-closed set in the space $X$. Suppose $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open. Then the definition of $\tilde{\delta}\omega$ -closed set in ideal topological space.

Example 3.1: Let $X = \{b, c, d\}$ be with topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$ and $\tau^c = \{\phi, \{b, d\}, \{b, c\}, \{b\}, X\}$. Then $\sigma cl(A)$ is $\tilde{\delta}\omega$ -closed set.

Theorem 3.2

Every closed set in $X$ is $\tilde{\delta}\omega$ -closed set in $X$.

Proof: Let $A$ be an arbitrary regular closed set in the space $X$, every closed set is g-closed set and from the theorem 3.1, every g-closed set in $X$ is $\tilde{\delta}\omega$ -closed. Thus every closed set in $X$ is $\tilde{\delta}\omega$ -closed.

The converse of the above theorem is not true, which is verified from the following example.

Example 3.2: Let $X = \{b, c, d\}$ be with the topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$ and $\tau^c = \{\phi, \{b, d\}, \{b, c\}, \{b\}, X\}$. Thus, the closed set is $\{\phi, \{b, d\}, \{b, c\}, \{b\}, X\}$. Here $A = \{c\}$ is $\tilde{\delta}\omega$ -closed set but not g-closed.

Theorem 3.3

Every regular closed set in $X$ is $\tilde{\delta}\omega$ -closed.

Proof: Let $A$ be an arbitrary regular closed set in the space $X$, every regular closed set is closed and from the theorem 3.1, every g-closed set in $X$ is $\tilde{\delta}\omega$ -closed. This implies every regular closed set in $X$ is $\tilde{\delta}\omega$ -closed.

The converse of the above theorem is not true, which is verified from the following example.

Example 3.3: Let $X = \{b, c, d\}$ be with the topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$ and $\tau^c = \{\phi, \{b, d\}, \{b, c\}, \{b\}, X\}$, the regular closed set, Hence, the arbitrary element $A$ is $\tilde{\delta}\omega$ -closed set but not regular closed.

Theorem 3.4

Every regular generalized closed set in $X$ is $\tilde{\delta}\omega$ -closed.

Proof: Let $A$ be an arbitrary regular generalized closed set in the space $X$. Suppose $cl(A) \subseteq U$. Whenever $A \subseteq U$ and $U$ is regular open. Then by the definition of $\tilde{\delta}\omega$ -closed set, if $\sigma cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $\omega$ open. Hence, the arbitrary element $A$ of regular generalized closed set belongs to $U$ and the arbitrary element $A$ of $\tilde{\delta}\omega$ -closed set belongs to $U$. This implies that $A$ is a $\tilde{\delta}\omega$ -closed set.

The converse of above theorem need not be true, which is verified from the following example.

Example 3.4: Let $X = \{b, c, d\}$ be with topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$ and $\tau^c = \{\phi, \{b, d\}, \{b, c\}, \{b\}, X\}$. Then $cl(A)$ is $\tilde{\delta}\omega$ -closed set but not regular generalized closed set.

Theorem 3.5

Every weakly generalized closed set in $X$ is $\tilde{\delta}\omega$ -closed.

Proof: Let $A$ be an arbitrary weakly generalized closed set in the space $X$. Then by definition of weakly generalized closed set and $\tilde{\delta}\omega$ -closed set the arbitrary element $A$ of weakly generalized closed set belongs to $U$ and the arbitrary element $A$ of $\tilde{\delta}\omega$ -closed set belongs to $U$. This implies that $A$ is a $\tilde{\delta}\omega$ -closed set.

The converse of the above theorem need not to be true, which is verified from the following example.

Example 3.5: Let $X = \{b, c, d\}$ be with the topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$. Now, if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$. Then $\tilde{\delta}\omega$ -closed set will be $\{\phi, \{b\}, \{b, d\}, \{b\}, X\}$. Here $A = \{c\}$ is a $\tilde{\delta}\omega$ -closed set but not weakly generalized closed.

Theorem 3.6

Every semi closed set in $X$ is $\tilde{\delta}\omega$ -closed.

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\(\delta\omega\) - closed sets in ideal topological space

Proof: Let \(A\) be an arbitrary semi closed set in the space \(X\), every semi closed set is closed and from the theorem 3.1 every closed set in \(X\) is \(\delta\omega\) -closed set. This implies, every semi closed set in \(X\) is \(\delta\omega\) closed set.

The converse of the above theorem is not true, which is verified from the following example.

Example 3.6: Let \(X = \{b, c, d\}\) be with the topology \(\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}\) Then from the definition of semi closed set \(\delta\omega\) -closed set belongs to \(U\), whenever \(A \subseteq U\) and \(U\) is semi open in \(X\) (Every weakly open set is open in \(X\)). Then by the definition of \(\delta\omega\) -closed set, if \(\sigma cl(A) \subseteq \phi\) and \(\delta\omega\) -closed set belongs to \(U\). Hence, the arbitrary element \(A\) of \(\delta\omega\) -closed set belongs to \(U\). This implies that \(A\) is a \(\delta\omega\) -closed set in \(X\).

Theorem 3.7
Every weekly closed set in \(X\) is \(\delta\omega\) -closed set.

Proof: Let \(A\) be a weekly closed set in the space \(X\). Suppose \(cl(A) \subseteq U\) When ever \(A \subseteq U\) and \(U\) is semi open in \(X\) . i.e., when ever \(A \subseteq U\) and \(U\) is semi open, every semi open set is open in \(X\). Then by the definition of \(\delta\omega\) -closed set, if \(\sigma cl(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is semi open in \(X\). Hence, the arbitrary element \(A\) of weakly closed set belongs to \(U\) and the arbitrary element \(A\) of \(\delta\omega\) -closed set belongs to \(U\). Thus implies that \(A\) is a \(\delta\omega\) -closed set in \(X\). The converse of the above theorem need not to be true, which is verified from the following example.

Example 3.7: Let \(X = \{b, c, d\}\) be with the topology \(\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}\). Now if \(cl(A) \subseteq \phi\), whenever \(A \subseteq U\) and \(U\) is semi open in \(X\). Then \(U = \{\phi, \{c\}, \{d\}, X\}\). This implies that \(A\) is a \(\delta\omega\) -closed set in \(X\).

Theorem 3.8
Every regular weakly generalized closed set in \(X\) is \(\delta\omega\) - closed.

Proof: Let \(A\) be a regular weakly generalized closed set in the space \(X\). Suppose \(cl(int(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\). i.e., \(A \subseteq U\) and \(U\) is open, every regular open set in \(X\) is open. Then by the definition of \(\delta\omega\) -closed set, if \(\sigma cl(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is \(\delta\omega\) open in \(X\). Hence, the arbitrary element \(A\) of regular weakly generalized closed set belongs to \(U\) and the arbitrary element \(A\) of \(\delta\omega\) -closed set belongs to \(U\). This implies that \(A\) is a \(\delta\omega\) -closed set in \(X\).

Example 3.8: Let \(X = \{b, c, d\}\) be with the topology \(\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}\) and \(\tau^c = \{\phi, \{b, d\}, \{b, c\}, \{b\}, X\}\). Then regular weakly generalized closed set belongs to \(U\). Hence, the arbitrary element \(A\) of \(\delta\omega\) -closed set belongs to \(U\). Thus we can say that This implies that \(A\) is a \(\delta\omega\) -closed set.

Theorem 3.9
Every regular semi closed set in \(X\) is \(\delta\omega\) - closed.

Proof: Let \(A\) be an arbitrary regular semi closed set in the space \(X\). Suppose \(U \subseteq U \subseteq A \subseteq cl(U)\) whenever \(U\) is regular open set in \(X\) is open. i.e., \(U\) is open. Then by the definition of \(\delta\omega\) -closed set, if \(\sigma cl(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is \(\delta\omega\) open in \(X\). Hence, the arbitrary element \(A\) of arbitrary regular semi closed set belongs to \(U\) and the arbitrary element \(A\) of \(\delta\omega\) -closed set belongs to \(U\). Thus we can say that This implies that \(A\) is a \(\delta\omega\) -closed set.

Theorem 3.10
Every regular weakly closed set in \(X\) is \(\delta\omega\) - closed.

Proof: Let \(A\) be an arbitrary regular weakly closed set in the space \(X\), every semi open set is open and from the theorem 3.2, every closed set in \(X\) is \(\delta\omega\) -closed set. This implies that every regular weakly closed set in \(X\) is \(\delta\omega\) -closed set. The converse of the above theorem is not true, which is verified using following example.

Example 3.10: Let \(X = \{b, c, d\}\) be with the topology \(\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}\). Then by the definition of regular weakly closed set \(\delta\omega\) -closed set is \(\{\phi, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, X\}\). Here \(A = \{c, d\}\) is \(\delta\omega\) -closed set but not regular weakly closed.

Theorem 3.11
Every \(*g\) - closed set in \(X\) is \(\delta\omega\) - closed.

Proof: Let \(A\) be an arbitrary \(*g\) - closed set in the space \(X\). Suppose \(cl(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is semi open \(X\). i.e., whenever \(A \subseteq U\) and \(U\) is semi open, every weakly open set is open in \(X\). Then by the definition of \(\delta\omega\) -closed set, if \(\sigma cl(A) \subseteq\)
U, whenever A ⊆ U and U is ω open in X. Hence, the arbitrary element A of *g-closed set belongs to U and the arbitrary element A of δω-closed set belongs to U. This implies that A of δω - closed set in X.

The converse of the above theorem need not to be true, which is verified using following example.

Example 3.11: Let X = {b, c, d} be with the topology τ = {∅, {c}, {d}, {c, d}, X}. Now if cl(A) ⊆ U, whenever A ⊆ U and U is weakly open in X. Then U = {∅, {c}, {d}, {c, d}, X}. Then *g-closed set will be {∅, {b}, {b, d}, {b, c}, X}. Here A = {c} is a δω closed set, but not *g-closed set in X.

Theorem 3.12

Every ω-closed set in X is δω – closed set.

Proof: Let A be an arbitrary ω-closed set in space X. Suppose clω(A) ⊆ U, whenever A ⊆ U and U is open. i.e., A ⊆ U and U is open. Then by the definition of δω-closed set, if clω(A) ⊆ U , whenever A ⊆ U and U is δω open in X. Hence, the arbitrary element A of ω-closed set belongs to U and also the arbitrary element A of δω-closed set belongs to U. This implies that, A is δω closed.

The converse of the above theorem is not true, which is verified from the following example.

Example 3.12: Let X = {b, c, d} be with the topology τ = {∅, {c}, {d}, {c, d}}, X}. Now if clθ(A) ⊆ U, whenever A ⊆ U and U is open in X. Then *θ-closed set will be {∅, {b, c}, {b, d}, {b}}. Here A = {c} is a δω-closed set, but not *θ-closed set.

Theorem 3.13

Every δ-closed set in X is δω - closed set in X.

Proof: Let A be an arbitrary δ-closed set in space X. Suppose clδ(A) ⊆ U, whenever A ⊆ U and U is open. i.e., A ⊆ U and U is open. Then by the definition δω-closed set, if clδ(A) ⊆ U , whenever A ⊆ U and U is δω open in X. Hence, the arbitrary element A of δ-closed set belongs to U and also the arbitrary element A of δω-closed set belongs to U. This implies that, A is δω closed.

The converse of the above theorem is not true, which is verified from the following example.

Example 3.13: Let X = {b, c, d} be with the topology τ = {∅, {c}, {d}, {c, d}, X}. Now if clδ(A) ⊆ U, whenever A ⊆ U and U is open in X. Then δ-closed set will be {∅, {b, c}, {b, d}, {b}}. Here A = {c} is a δω-closed set, but not δ-closed.

IV. SOME OPERATIONS ON δω - CLOSED SETS

Theorem 4.1

The union of two δω-closed sets of X is also an δω-closed sets of X.

Proof: Assume that A and B are δω-closed set in X. Let U be open in X, such that A ∪ B ⊆ U. Thus A ⊆ U and B ⊆ U. Since A and B are δω-closed set so clδ(A) ⊆ U and clδ(B) ⊆ U. Hence clδ(A ∪ B) = clδ(A) ∪ clδ(B) ⊆ U. i.e., clδ(A ∪ B) ⊆ U. Hence A ∪ B is an δω-closed set in X.

Theorem 4.2

If a subset A of X is δω-closed in X, then clδ(A)\A, A does not contain any non-empty open set in X.

Proof: Suppose that A is δω-closed set in X. Let U be open set such that clδ(A)\A ⊆ U and U ≠ ∅. Now U ⊆ clδ(A)\A, i.e., U ⊆ X\A which implies that A ⊆ X\U. As U is open, X\U is also open in X. Since A is an δω-closed set in ideal topological space, X\U is a δω-closed set in X. So U ⊆ X\σclδ(A). Therefore U ⊆ clδ(A) ∩ (X\σclδ(A)) = ∅. This shows that U = ∅, which is contradiction. Hence clδ(A)\A does not contain any non-empty open set in X.

Theorem 4.3

For an element x ∈ X, the set X\{x} is δω-closed or ω-open.

Proof: Let x ∈ X. Suppose X\{x} is not ω-open. Then X is the only ω-open set containing X\{x} ,which means that the only choice of ω-open set containing X\{x} in X. i.e., X\{x} ⊆ X. Also, we know X\{x} is not δω-closed. To prove X\{x} is open. Suppose X\{x} is not open. As X\{x} is a subset of x and X\{x} only but X\{x} is not open. Thus the only open set in X. Also σcl(X\{x}) ⊆ X. Therefore by the definition of δω-closed sets X\{x} is δω-closed, which is a contradiction. Hence X\{x} is ω-open.

Theorem 4.4
If $A$ is an $\delta\omega$-closed subset of $X$ such that $A \subset B \subset \sigmacl(A)$, then $B$ is an $\delta\omega$-closed set in $X$.

Proof: Let $A$ be an $\delta\omega$-closed set in $X$ such that $A \subset B \subset \sigmacl(A)$. Let $U$ be open set such that $B \subset U$, then $A \subset U$. Since $A$ is $\delta\omega$-closed, we have $\sigmacl(A) \subset U$. Now as $B \subset \sigmacl(A)$, $\sigmacl(B) \subset \sigmacl(\sigmacl(A)) \subset \sigmacl(A) \subset U$. Thus $\sigmacl(B) \subset U$, whenever $B \subset c$ and $U$ is open. Therefore $B$ is an $\delta\omega$-closed set in $X$.

Converse of the theorem is not true, which is verified from the following example.

Example 4.2: Let $X = \{b, c, d\}$ be with the topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$ $\delta\omega$-closed set is $\{c, d\}$ which is contained in each $\omega$ open set. $\sigmacl(B) = \{c\}$, which is also contained in each open set. Thus by the definition, $A$ and $B$ both are $\delta\omega$-closed.

Theorem 4.5

If a subset $A$ of a topological space $X$ is both $\omega$ open and $\delta\omega$-closed then it is $\omega$ closed.

Proof: Suppose a subset $A$ of a topological space in $X$ is both $\omega$ open and $\delta\omega$-closed. Now $A \subset A$ then by definition of $\delta\omega$-closed we have $\sigmacl(A) \subset A$. So $A \subset \sigmacl(A)$. Thus we have $\sigmacl(A) = \sigmacl(A)$. Finally $A$ is open.

Theorem 4.6

If a subset $A$ of a topological space $X$ is both open and $\delta\omega$-closed then it is closed.

Proof: Suppose a subset $A$ of a topological space in $X$ is both open and $\delta\omega$-closed. Now $A \subset A$ then by definition of $\delta\omega$-closed we have $\sigmacl(A) \subset A$. So $A \subset \sigmacl(A)$. Thus we have $\sigmacl(A) = \sigmacl(A)$. Finally $A$ is open.

Theorem 4.7

In a topological space $X$ if open of $X$ are $\{X, \phi\}$, then every subset of $X$ is an $\delta\omega$-closed set.

Proof: Let $X$ be topological space and open. i.e., $\{X, \phi\}$. Suppose $A$ be any arbitrary subset of $X$, if $A = \phi$ then $X$ is an $\delta\omega$-closed set in $X$. If $A \neq \phi$ then $X$ is the only open set containing $A$ and so $\sigmacl(A) \subset X$. Hence by the definition $A$ is $\delta\omega$-closed in $X$. The converse is not true, which is verified by following example.

Example 4.2: Let $X = \{b, c, d\}$ be with the topology $\tau = \{\phi, \{c\}, \{d\}, \{c, d\}, X\}$, every subset of $X$ is an $\delta\omega$-closed set in $X$. Thus for every subset of $X$ is an $\delta\omega$-closed set, we need not be open set only if $\{X, \phi\}$.

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