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A Comprehensive Comparative Study of Two-Dimensional and Three-Dimensional Fractional Fourier Transforms

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Abstract: The Fractional Fourier Transform (FrFT) has established itself as a fundamental mathematical tool in signal processing, optics, and communications, with applications spanning filter design, signal recovery, phase retrieval, pattern recognition, and image encryption. While the one-dimensional FrFT is theoretically well-understood, modern applications increasingly involve multidimensional data, particularly images (2D) and medical/geophysical volumes (3D). This paper presents a comprehensive comparative study of separable two-dimensional and three-dimensional Fractional Fourier Transforms. We provide complete mathematical proofs of fundamental properties including linearity, index additivity, shift and modulation theorems, Parseval's relation, eigenfunction expansions, and convolution structures for both 2D and 3D cases. Through systematic comparison, we demonstrate that while separable FrFTs preserve many desirable properties in higher dimensions, they are inherently limited to separable phase space rotations. We establish the relationship between separable multidimensional FrFTs and more general non-separable linear canonical transforms, identifying scenarios where separable transforms suffice and where genuinely coupled transforms become necessary.

Keywords: Fractional Fourier Transform, Multidimensional Signal Processing, Separable Transforms, Phase Space Analysis, Hermite-Gaussian Functions, Linear Canonical Transforms

I. INTRODUCTION

The Fractional Fourier Transform (FrFT) generalizes the classical Fourier transform by introducing a continuous order parameter corresponding to rotation in the time–frequency plane [1]–[3]. While the one-dimensional FrFT is well established, modern scientific and engineering problems frequently involve multidimensional signals. In particular, two-dimensional FrFTs arise in image processing and optics, whereas three-dimensional FrFTs appear in medical imaging, geophysics, and volumetric data analysis. This paper addresses critical gaps in the literature through three primary contributions:

1) Complete mathematical proofs of fundamental properties for both 2D and 3D separable FrFTs.
2) Explicit identification of the separable phase space rotation property and its geometric interpretation.
3) Comparative analysis establishing the relationship between separable multidimensional FrFTs and non-separable linear canonical transforms.

A. Mathematical Preliminaries

1) One-Dimensional Fractional Fourier Transform

Definition 1 (1D Fractional Fourier Transform). *For a function $f(x) \in L^2(\mathbb{R})$, the FrFT of order α (with associated angle $\theta = \alpha\pi/2$) is defined as:*

$$\mathcal{F}^{\alpha}[f(x)](u) = F_{\alpha}(u) = \int_{-\infty}^{\infty} f(x)\, K_{\alpha}(x,u)\, dx,$$

where the kernel $K_{\alpha}(x,u)$ is given by

$$K_{\alpha}(x,u) = \begin{cases} A_{\theta}\exp\left[i\left(\frac{x^2+u^2}{2}\cot\theta - xu\csc\theta\right)\right], & \theta \neq n\pi, \\ \delta(x-u), & \theta = 2n\pi, \\ \delta(x+u), & \theta = (2n+1)\pi, \end{cases}$$

with amplitude factor $A_{\theta} = \sqrt{\frac{1-i\cot\theta}{2\pi}}$.

The kernel satisfies the index additivity (semigroup) property:

$$\int_{-\infty}^{\infty} K_\alpha(x,u')\,K_\beta(u',u)\,du' = K_{\alpha+\beta}(x,u),$$

and the kernel conjugation property:

$$\overline{K_\alpha(x,u)} = K_{-\alpha}(x,u).$$

2) Separable Multidimensional Fractional Fourier Transforms

Definition 2 (2D Separable FrFT). *For $f(x,y) \in L^2(\mathbb{R}^2)$ with fractional orders α, β (angles $\theta = \alpha\pi/2$, $\phi = \beta\pi/2$)*

$\mathcal{F}^{\alpha,\beta}[f(x,y)](u,v) = \iint_{-\infty}^{\infty} f(x,y)\,K_\alpha(x,u)\,K_\beta(y,v)\,dx\,dy.$

Definition 3 (3D Separable FrFT). *For $f(x,y,z) \in L^2(\mathbb{R}^3)$ with fractional orders α, β, γ (angles $\theta = \alpha\pi/2$, $\phi = \beta\pi/2$, $\psi = \gamma\pi/2$)*

$\mathcal{F}^{\alpha,\beta,\gamma}[f(x,y,z)](u,v,w) = \iiint_{-\infty}^{\infty} f(x,y,z)\,K_\alpha(x,u)\,K_\beta(y,v)\,K_\gamma(z,w)\,dx\,dy\,dz.$

B. Comparative Properties and Proofs

1) Linearity

Theorem 4 (2D Linearity). *For constants $a, b \in \mathbb{C}$ and functions $f, g \in L^2(\mathbb{R}^2)$:*

$\mathcal{F}^{\alpha,\beta}[af(x,y) + bg(x,y)] = a\,\mathcal{F}^{\alpha,\beta}[f(x,y)] + b\,\mathcal{F}^{\alpha,\beta}[g(x,y)].$

Proof.

$$\begin{aligned}\mathcal{F}^{\alpha,\beta}[af+bg](u,v) &= \iint [af(x,y)+bg(x,y)]\,K_\alpha(x,u)\,K_\beta(y,v)\,dx\,dy\\ &= a\iint f(x,y)\,K_\alpha(x,u)\,K_\beta(y,v)\,dx\,dy\\ &\quad + b\iint g(x,y)\,K_\alpha(x,u)\,K_\beta(y,v)\,dx\,dy\\ &= a\,F_{\alpha,\beta}(u,v) + b\,G_{\alpha,\beta}(u,v).\end{aligned}$$

□

Theorem 5 (3D Linearity). *For constants $a, b \in \mathbb{C}$ and functions $f, g \in L^2(\mathbb{R}^3)$:*

$\mathcal{F}^{\alpha,\beta,\gamma}[af + bg] = a\,\mathcal{F}^{\alpha,\beta,\gamma}[f] + b\,\mathcal{F}^{\alpha,\beta,\gamma}[g].$

Proof. The proof follows the same pattern as the 2D case with an additional integration dimension, splitting the triple integral by linearity. □

2) Index Additivity

Theorem 6 (2D Index Additivity). $\mathcal{F}^{\alpha_2,\beta_2}\{\mathcal{F}^{\alpha_1,\beta_1}[f(x,y)]\} = \mathcal{F}^{\alpha_1+\alpha_2,\,\beta_1+\beta_2}[f(x,y)].$

Proof. Let $G(u',v') = \mathcal{F}^{\alpha_1,\beta_1}[f](u',v')$. Then:

$$\begin{aligned}\mathcal{F}^{\alpha_2,\beta_2}[G](u,v) &= \iint G(u',v')\,K_{\alpha_2}(u',u)\,K_{\beta_2}(v',v)\,du'\,dv'\\ &= \iiiint f(x,y)\,K_{\alpha_1}(x,u')\,K_{\beta_1}(y,v')\,K_{\alpha_2}(u',u)\,K_{\beta_2}(v',v)\,dx\,dy\,du'\,dv'.\end{aligned}$$

By Fubini's theorem, interchange integration order:

$$= \iint f(x,y)\Big[\int K_{\alpha_1}(x,u')\,K_{\alpha_2}(u',u)\,du'\Big]\Big[\int K_{\beta_1}(y,v')\,K_{\beta_2}(v',v)\,dv'\Big]dx\,dy.$$

Applying the 1D semigroup property $\int K_a(x,u')K_b(u',u)\,du' = K_{a+b}(x,u)$:

$$= \iint f(x,y)\,K_{\alpha_1+\alpha_2}(x,u)\,K_{\beta_1+\beta_2}(y,v)\,dx\,dy = \mathcal{F}^{\alpha_1+\alpha_2,\,\beta_1+\beta_2}[f](u,v).$$

□

Theorem 7 (3D Index Additivity). $\mathcal{F}^{\alpha_2,\beta_2,\gamma_2}\{\mathcal{F}^{\alpha_1,\beta_1,\gamma_1}[f]\} = \mathcal{F}^{\alpha_1+\alpha_2,\,\beta_1+\beta_2,\,\gamma_1+\gamma_2}[f(x,y,z)].$

Proof. Interchanging integration order via Fubini's theorem and applying the 1D semigroup property to each of the three dimensions yields the result. □

3) Shift Theorem

Theorem 8 (2D Shift Theorem). *Let $\theta = \alpha\pi/2$, $\phi = \beta\pi/2$. Then:*

$$\begin{aligned}&\mathcal{F}^{\alpha,\beta}[f(x-x_0,\,y-y_0)](u,v)\\ &\quad = \exp[i/2\,x_0^2\sin\theta\cos\theta - iux_0\sin\theta]\exp[i/2\,y_0^2\sin\phi\cos\phi - ivy_0\sin\phi]\\ &\qquad \times F_{\alpha,\beta}(u - x_0\cos\theta,\ v - y_0\cos\phi).\end{aligned}$$

Proof. Substitute $s = x - x_0$. The kernel satisfies the shifting identity:

$$K_\alpha(s + x_0, u) = K_\alpha(s, u - x_0\cos\theta) \cdot \exp[-iux_0\sin\theta + i/2\, x_0^2\sin\theta\cos\theta].$$

Substituting and factoring the exponential out of the integral gives $F_{\alpha,\beta}(u - x_0\cos\theta, v)$ multiplied by the phase factor. Applying the analogous result for the y-shift yields the complete formula. □

Theorem 9 (3D Shift Theorem). *Let $\theta = \alpha\pi/2, \phi = \beta\pi/2, \psi = \gamma\pi/2$. Then:*

$$\begin{aligned}\mathcal{F}^{\alpha,\beta,\gamma}&[f(x - x_0, y - y_0, z - z_0)](u, v, w)\\ &= \exp[i/2\, x_0^2\sin\theta\cos\theta - iux_0\sin\theta]\exp[i/2\, y_0^2\sin\phi\cos\phi - ivy_0\sin\phi]\\ &\quad\times \exp[i/2\, z_0^2\sin\psi\cos\psi - iwz_0\sin\psi]\\ &\quad\times F_{\alpha,\beta,\gamma}(u - x_0\cos\theta,\ v - y_0\cos\phi,\ w - z_0\cos\psi).\end{aligned}$$

Proof. Apply the kernel shifting identity to each dimension independently and factor the exponential terms. □

4) Modulation Theorem

Theorem 10 (2D Modulation Theorem). *Let $\theta = \alpha\pi/2, \phi = \beta\pi/2$. Then:*

$$\begin{aligned}\mathcal{F}^{\alpha,\beta}&[e^{i\omega_1 x}e^{i\omega_2 y}f(x, y)](u, v)\\ &= \exp[-i/2\, \omega_1^2\sin\theta\cos\theta + iu\omega_1\cos\theta]\exp[-i/2\, \omega_2^2\sin\phi\cos\phi + iv\omega_2\cos\phi]\\ &\quad\times F_{\alpha,\beta}(u - \omega_1\sin\theta,\ v - \omega_2\sin\phi).\end{aligned}$$

Proof. The kernel satisfies the modulation identity:

$$e^{i\omega_1 x}K_\alpha(x, u) = K_\alpha(x, u - \omega_1\sin\theta) \cdot \exp[iu\omega_1\cos\theta - i/2\, \omega_1^2\sin\theta\cos\theta].$$

Substituting, factoring the exponential out, and combining with the analogous y-modulation result completes the proof. □

Theorem 11 (3D Modulation Theorem). *Let $\theta = \alpha\pi/2, \phi = \beta\pi/2, \psi = \gamma\pi/2$. Then:*

$$\begin{aligned}\mathcal{F}^{\alpha,\beta,\gamma}&[e^{i\omega_1 x}e^{i\omega_2 y}e^{i\omega_3 z}f(x, y, z)](u, v, w)\\ &= \exp[-i/2\, \omega_1^2\sin\theta\cos\theta + iu\omega_1\cos\theta]\exp[-i/2\, \omega_2^2\sin\phi\cos\phi + iv\omega_2\cos\phi]\\ &\quad\times \exp[-i/2\, \omega_3^2\sin\psi\cos\psi + iw\omega_3\cos\psi]\\ &\quad\times F_{\alpha,\beta,\gamma}(u - \omega_1\sin\theta,\ v - \omega_2\sin\phi,\ w - \omega_3\sin\psi).\end{aligned}$$

Proof. Apply the modulation identity to each dimension independently; exponential factors combine multiplicatively. □

5) Parseval's Theorem

Theorem 12 (2D Parseval's Theorem). *For any $f, g \in L^2(\mathbb{R}^2)$: $\iint_{-\infty}^{\infty} f(x, y)\,\overline{g(x, y)}\,dx\,dy = \iint_{-\infty}^{\infty} F_{\alpha,\beta}(u, v)\,\overline{G_{\alpha,\beta}(u, v)}\,du\,dv$. In particular, energy is conserved: $\iint |f(x, y)|^2\,dx\,dy = \iint \left|F_{\alpha,\beta}(u, v)\right|^2\,du\,dv$.*

Proof. Expand the right-hand side using the kernel definitions and the conjugation property $\overline{K_\alpha(x, u)} = K_{-\alpha}(x, u)$. The inner u-integral satisfies the orthogonality condition:

$$\int K_\alpha(x, u)\,K_{-\alpha}(x', u)\,du = \delta(x - x'),$$

and similarly for v. Applying both yields $\iint f(x, y)\,\overline{g(x, y)}\,dx\,dy$. □

Theorem 13 (3D Parseval's Theorem). *For any $f, g \in L^2(\mathbb{R}^3)$:*
$\iiint f(x, y, z)\,\overline{g(x, y, z)}\,dx\,dy\,dz = \iiint F_{\alpha,\beta,\gamma}(u, v, w)\,\overline{G_{\alpha,\beta,\gamma}(u, v, w)}\,du\,dv\,dw$.

Proof. The proof mirrors the 2D case, applying the kernel orthogonality condition in all three dimensions. □

6) Generalized Convolution Theorems

Definition 14 (2D Generalized Convolution). *For $\theta = \alpha\pi/2, \phi = \beta\pi/2$, define: $(f *_{\alpha,\beta} g)(x, y) = \iint f(x', y')\,g(x - x', y - y')\,e^{i\Phi(x,x',\theta)}\,e^{i\Phi(y,y',\phi)}\,dx'\,dy'$, where the phase function is: $\Phi(a, b, \theta) = \frac{1}{2}[b^2 + (a - b)^2 - a^2]\cot\theta$.*

Theorem 15 (2D Convolution Theorem). *Let $\tilde{F}_{\alpha,\beta}(u, v) = e^{\frac{i}{2}u^2\cot\theta}\,e^{\frac{i}{2}v^2\cot\phi}\,F_{\alpha,\beta}(u, v)$ and similarly $\tilde{G}_{\alpha,\beta}$. Then:*
$\mathcal{F}^{\alpha,\beta}[(f *_{\alpha,\beta} g)(x, y)](u, v) = C_\theta\,C_\phi\,e^{-\frac{i}{2}u^2\cot\theta}\,e^{-\frac{i}{2}v^2\cot\phi}\,\tilde{F}_{\alpha,\beta}(u, v)\,\tilde{G}_{\alpha,\beta}(u, v)$, *where $C_\theta = \sqrt{1 - i\cot\theta}$.*

Proof. Substitute $H(x, y) = (f *_{\alpha,\beta} g)(x, y)$. Change variables $s = x - x'$, $t = y - y'$ and apply the kernel shift and modulation properties. After algebraic manipulation using trigonometric identities, the integral separates into the product $\tilde{F}_{\alpha,\beta}(u, v) \cdot \tilde{G}_{\alpha,\beta}(u, v)$ multiplied by $C_\theta C_\phi e^{-\frac{i}{2}u^2\cot\theta}e^{-\frac{i}{2}v^2\cot\phi}$. □

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Definition 16 (3D Generalized Convolution). *For* $\theta = \alpha\pi/2, \phi = \beta\pi/2, \psi = \gamma\pi/2$: $(f *_{\alpha,\beta,\gamma} g)(x,y,z) = \iiint f(x',y',z')\, g(x-x', y-y', z-z')\, e^{i\Phi(x,x',\theta)}\, e^{i\Phi(y,y',\phi)}\, e^{i\Phi(z,z',\psi)}\, dx'\, dy'\, dz'$.

Theorem 17 (3D Convolution Theorem). *With* $\tilde{F}_{\alpha,\beta,\gamma}(u,v,w) = e^{\frac{i}{2}u^2\cot\theta}\, e^{\frac{i}{2}v^2\cot\phi}\, e^{\frac{i}{2}w^2\cot\psi}\, F_{\alpha,\beta,\gamma}(u,v,w)$:

$$\mathcal{F}^{\alpha,\beta,\gamma}[(f *_{\alpha,\beta,\gamma} g)(x,y,z)](u,v,w) = C_\theta\, C_\phi\, C_\psi\, e^{-\frac{i}{2}u^2\cot\theta}\, e^{-\frac{i}{2}v^2\cot\phi}\, e^{-\frac{i}{2}w^2\cot\psi}\, \tilde{F}_{\alpha,\beta,\gamma}(u,v,w)\, \tilde{G}_{\alpha,\beta,\gamma}(u,v,w).$$

Proof. Follows the same pattern as the 2D case with additional z-integration; phase compensation factors multiply across all three dimensions. □

7) Eigenfunction Property and Hermite-Gaussian Expansions

Definition 18 (1D Hermite-Gaussian Functions). $\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x)\, e^{-x^2/2}, \quad n = 0,1,2,\ldots$, *where* $H_n(x)$ *are the Hermite polynomials. These satisfy the eigenvalue equation:* $\mathcal{F}^\alpha[\psi_n(x)](u) = e^{-in\alpha\pi/2}\, \psi_n(u)$.

Theorem 19 (2D Eigenfunction Property). *The 2D Hermite-Gaussian functions* $\psi_{m,n}(x,y) = \psi_m(x)\psi_n(y)$, $\psi_{m,n}(x,y) = \frac{1}{\sqrt{2^{m+n} m!\, n!\, \pi}} H_m(x)\, H_n(y)\, e^{-(x^2+y^2)/2}$, *are eigenfunctions of the 2D separable FrFT:* $\mathcal{F}^{\alpha,\beta}[\psi_{m,n}(x,y)](u,v) = e^{-i(m\alpha+n\beta)\pi/2}\, \psi_{m,n}(u,v)$, *with eigenvalues* $\lambda_{m,n} = e^{-i(m\alpha+n\beta)\pi/2}$.

Proof. By separability of both the transform and the eigenfunction:

$$\begin{aligned}\mathcal{F}^{\alpha,\beta}[\psi_{m,n}](u,v) &= \int \psi_m(x) K_\alpha(x,u)\, dx \cdot \int \psi_n(y) K_\beta(y,v)\, dy \\ &= \left(e^{-im\alpha\pi/2}\psi_m(u)\right)\left(e^{-in\beta\pi/2}\psi_n(v)\right) = e^{-i(m\alpha+n\beta)\pi/2}\psi_{m,n}(u,v).\end{aligned}$$

□

Theorem 20 (3D Eigenfunction Property). *The 3D Hermite-Gaussian functions* $\psi_{m,n,p}(x,y,z) = \psi_m(x)\psi_n(y)\psi_p(z)$, $\psi_{m,n,p}(x,y,z) = \frac{1}{\sqrt{2^{m+n+p} m!\, n!\, p!\, \pi^{3/2}}} H_m(x)\, H_n(y)\, H_p(z)\, e^{-(x^2+y^2+z^2)/2}$, *are eigenfunctions of the 3D separable FrFT:* $\mathcal{F}^{\alpha,\beta,\gamma}[\psi_{m,n,p}](u,v,w) = e^{-i(m\alpha+n\beta+p\gamma)\pi/2}\, \psi_{m,n,p}(u,v,w)$, *with eigenvalues* $\lambda_{m,n,p} = e^{-i(m\alpha+n\beta+p\gamma)\pi/2}$.

Proof. By separability, the triple integral factors into three independent 1D FrFTs; apply the 1D eigenvalue equation to each factor. □

Theorem 21 (2D Eigenfunction Expansion). *Any* $f(x,y) \in L^2(\mathbb{R}^2)$ *can be expanded as:*

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{m,n}\, \psi_{m,n}(x,y), \quad c_{m,n} = \iint f(x,y)\, \psi_{m,n}(x,y)\, dx\, dy.$$

The 2D FrFT acts on this expansion as: $F_{\alpha,\beta}(u,v) = \sum_{m,n} c_{m,n}\, e^{-i(m\alpha+n\beta)\pi/2}\, \psi_{m,n}(u,v)$.

Theorem 22 (3D Eigenfunction Expansion). *Any* $f(x,y,z) \in L^2(\mathbb{R}^3)$ *can be expanded as:* $f(x,y,z) = \sum_{m,n,p} c_{m,n,p}\, \psi_{m,n,p}(x,y,z)$, *and the 3D FrFT acts as:* $F_{\alpha,\beta,\gamma}(u,v,w) = \sum_{m,n,p} c_{m,n,p}\, e^{-i(m\alpha+n\beta+p\gamma)\pi/2}\, \psi_{m,n,p}(u,v,w)$.

8) Isotropic Special Cases

Corollary 23 (Isotropic 2D FrFT). *For* $\alpha = \beta = a$, *eigenvalues depend only on total order* $N = m + n$: $\lambda_{m,n} = e^{-i(m+n)a\pi/2} = e^{-iNa\pi/2}$. *All eigenfunctions with the same total order* N *are degenerate, with degeneracy* $N + 1$.

Corollary 24 (Isotropic 3D FrFT). *For* $\alpha = \beta = \gamma = a$, *eigenvalues depend only on total order* $N = m + n + p$: $\lambda_{m,n,p} = e^{-i(m+n+p)a\pi/2} = e^{-iNa\pi/2}$. *The degeneracy for a given total order* N *is* $(N+1)(N+2)/2$.

C. Comparative Summary and Discussion

1) Systematic Property Comparison

Table 1 presents a comprehensive comparison of properties for 2D and 3D separable FrFTs.

Comprehensive Property Comparison of 2D and 3D Separable FrFTs

Property	2D	3D	Justification / Limitation
Linearity	✓	✓	Linear integral definition; holds in any dimension
Index Additivity	✓	✓	1D semigroup property applied separably
Reduction to FT	✓	✓	Special case: all orders = 1

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Property	2D	3D	Justification / Limitation
Unitarity (Parseval)	✓	✓	Kernel orthogonality extends separably
Invertibility	✓	✓	Inverse = separable FrFT with negative orders
Shift Theorem	✓	✓	Phase factors remain separable per dimension
Modulation Theorem	✓	✓	Modulation acts independently per dimension
Separable Eigenfunctions	✓	✓	Products of 1D HG functions form complete basis
Convolution Theorem	✓*	✓*	Requires generalized def. with phase factors
Phase Space Rotation	✓†	✓†	Only separable rotations; no coupling
Coupled Rotation	×	×	Requires non-separable LCT or NSFrFT
General Eigenfunctions	×	×	No angular momentum states
General Convolution	×	×	Standard convolution doesn't diagonalize

*Requires generalized convolution with quadratic phase factors.
†Separable rotations only.

2) Analysis of Extending Properties

a) Properties That Extend Fully

Properties that are purely algebraic or depend on the tensor product structure extend straightforwardly: linearity, index additivity, unitarity, and separable eigenfunctions.

b) Properties That Require Modification

The convolution operation requires a generalized definition with quadratic phase factors. Shift and modulation theorems acquire additional phase factors that remain separable but are algebraically more complex.

c) Properties That Do Not Extend

The separable FrFT cannot perform coupled phase space rotations. In 2D, the 4D Wigner distribution $W_f(x, y, \xi, \eta)$ transforms as:

$$W_{\mathcal{F}^{\alpha,\beta}[f]}(x, y, \xi, \eta) = W_f(x\cos\theta - \xi\sin\theta,\ y\cos\phi - \eta\sin\phi,\ x\sin\theta + \xi\cos\theta,\ y\sin\phi + \eta\cos\phi),$$

which cannot mix x with η or y with ξ.

D. Limitations and Relationship to Non-Separable Transforms

1) The Coupled Rotation Limitation

The separable FrFT corresponds to a block-diagonal symplectic matrix:

$$S_{\text{sep}} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{pmatrix},$$

whereas a general non-separable linear canonical transform allows fully populated off-diagonal blocks that couple all four phase-space dimensions.

2) Relationship to Linear Canonical Transforms

For 2D, the general LCT parameter matrix is $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

where A, B, C, D are 2×2 matrices satisfying symplecticity conditions. The separable FrFT corresponds to the special case where all four sub-matrices are diagonal.

Li et al. introduced a two-dimensional nonseparable FrFT (NSFrFT) with four degrees of freedom that includes the separable FrFT, gyrator transform, and coupled FrFT as special cases.

3) When Separable Transforms Suffice

Separable transforms are adequate for: image processing (filtering independently applied to rows and columns), separable physical systems, computational efficiency ($O(N^2 \log N)$ for 2D, $O(N^3 \log N)$ for 3D), and separable filter design.

4) When Non-Separable Transforms Are Necessary

Non-separable transforms are essential for: coupled physical systems (anisotropic optical media), signals with angular momentum or vortex structure, joint time-vertex processing, and full phase-space rotation of the 4D Wigner distribution.

II. CONCLUSIONS

This paper has presented a comprehensive comparative study of 2D and 3D separable Fractional Fourier Transforms, providing complete proofs of linearity, index additivity, shift and modulation theorems, Parseval's relation, eigenfunction expansions, and generalized convolution theorems.

Our systematic comparison confirms that separable multidimensional FrFTs successfully extend unitarity, invertibility, and complete eigenfunction expansions from the 1D case, but are fundamentally limited to block-diagonal symplectic (separable) phase-space rotations. For applications involving separable systems, the separable FrFT is an efficient and theoretically complete tool. When coupling between dimensions or full phase-space rotation is required, non-separable generalizations such as the NSFrFT or angular graph fractional transforms become necessary.

Future directions include: fast algorithms for non-separable multidimensional FrFTs, uncertainty principles in higher-dimensional fractional domains, adaptive fractional transforms with learnable parameters, extensions to graph-structured domains, and applications in quantum information processing.

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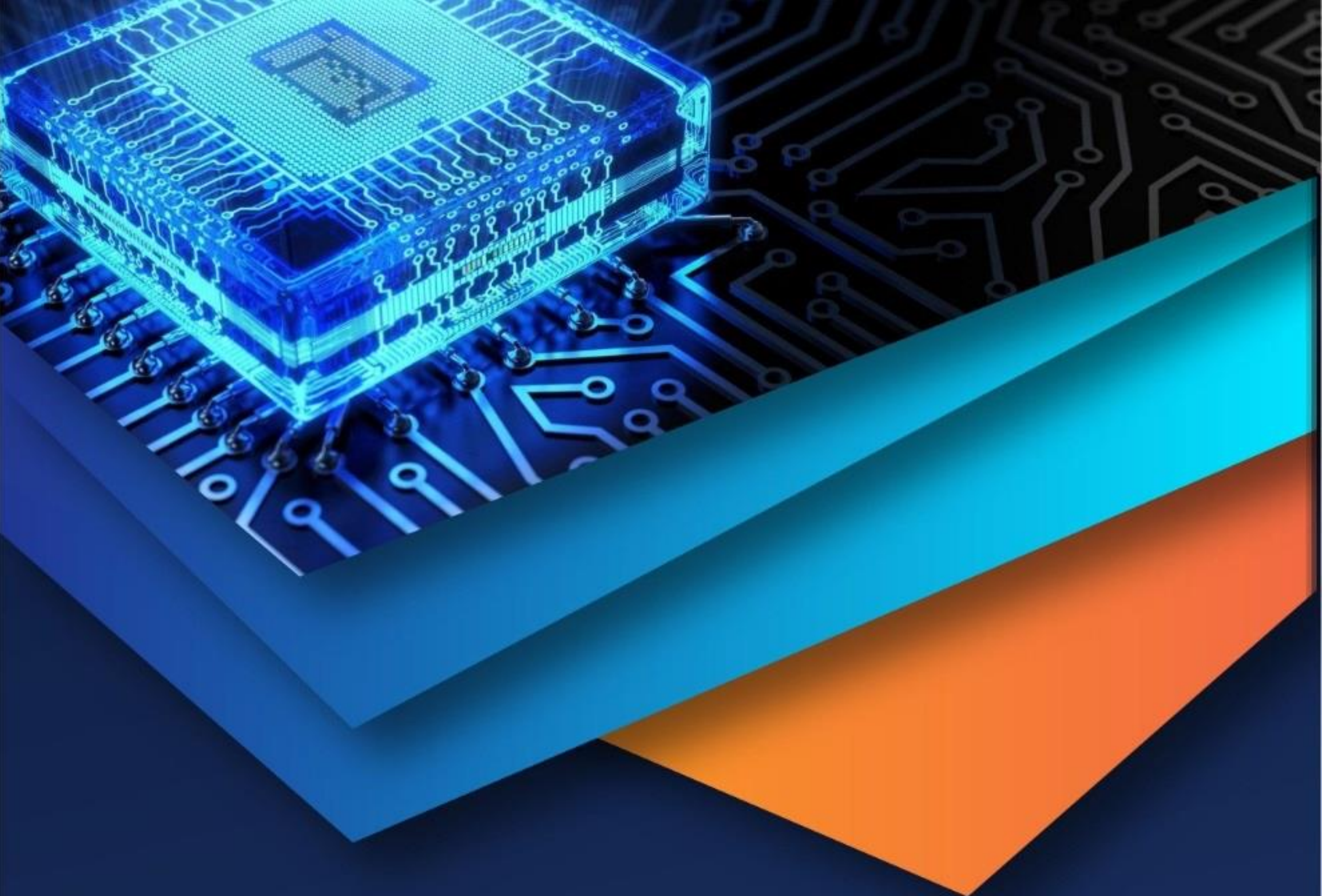

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ISRA JIF IMPACT FACTOR: 7.129

SJIFactor TOGETHER WE REACH THE GOAL IMPACT FACTOR: 7.429

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