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A Review on Kober's Trigonometric and Hyperbolic Inequalities in Optimization and Control Application

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Abstract: Kober's trigonometric and hyperbolic inequalities form a significant class of functional inequalities that provide sharp bounds for nonlinear expressions involving sine, cosine, hyperbolic sine, and hyperbolic cosine functions. These inequalities, originally established to refine classical analytic results, have found widespread application in modern optimization and control theory. In control system analysis, such inequalities are instrumental in deriving stability conditions, particularly in Lyapunov-based methods and the analysis of time-delay and nonlinear systems. Moreover, they serve as essential tools in the design of robust and adaptive controllers, where bounding nonlinearities is crucial for ensuring system performance and convergence. In optimization problems involving trigonometric constraints or cost functions, Kober-type inequalities aid in the convexification and approximation of non-convex terms, thereby enabling tractable solutions. This work presents an overview of Kober's inequalities involving trigonometric and hyperbolic functions and explores their theoretical foundations, refinements, and applications in control system design and constrained optimization.

Keywords: Kober trigonometric inequalities, Kober hyperbolic inequalities, Control system optimization, Time-delay system.

I. INTRODUCTION

Hermann Kober was a mathematician known for contributions to special functions, fractional calculus, and inequalities, particularly involving integral operators and functional estimates. The Kober inequalities typically refer to inequalities associated with Kober fractional integrals, but the name is also linked with several refined inequalities involving trigonometric and hyperbolic functions, often used in analysis and optimization problems.

Kober introduced several inequalities [1,2,3,4,5,6] involving functions like: sinx , cosx, tanx, sinhx , coshx, tanhx with comparisons to linear, power, or exponential functions. The Kober-type trigonometric inequality as below

$$\frac{\sin x}{x} < \cos\left(\frac{x}{2}\right) \quad for\ 0 < x < \pi \qquad \text{-----------(1)}$$

and Kober-type hyperbolic inequality as below

$$\frac{\sin hx}{x} < \cos h\left(\frac{x}{2}\right) \quad for\ \ x > 0 \qquad \_\_\_\_\_\_\_\_(2)$$

These inequalities are used to establish the bounds and approximations useful in control stability, error estimation, and convergence analysis. Kober-type inequalities help to estimate Lyapunov function behavior and useful in proving the stability of nonlinear and time-delay systems. In control system, Kober inequalities provide performance bounds that offer tight and tractable for system output and control the error.

Kober inequalities has wide importance in control system and optimization due to it provide the bounds for non linear functions, in design of controllers and gain tuning. In signal processing and filtering, Kober-type inequalities assist in **error bounds** that rely on sine or hyperbolic kernel functions. In optimization process, trigonometric constraints often appear (e.g. phase angles, trajectory planning) that we convexificate by kober-type bounds.

II. LITERATURE REVIEW

Kober, H. [2] (1957), **"**Über einige Mittelwerte mit Anwendungen auf Differential- und Integral gleichungen" Journal für die reine und angewandte Mathematik, Crelle's Journal, introduces important inequalities involving trigonometric and hyperbolic functions, laying the foundation for subsequent applications in analysis and control.

Boyd, S., Vandenberghe, L.[7] (2004) "Convex Optimization", explains how the inequalities, including trigonometric and hyperbolic bounds, are used in convex relaxation and control optimization problems.

Sándor, J. [8] (2011) "On Some Trigonometric and Hyperbolic Inequalities, "Journal of Inequalities in Pure and Applied Mathematics, explores refined trigonometric and hyperbolic inequalities including those related to Kober's results, relevant in nonlinear system analysis.

Klén, R., Vuorinen, M., Zhang, X.[9] (2013) "Inequalities for the Generalized Trigonometric and Hyperbolic Functions" Journal of Approximation Theory, extends classical inequalities including Kober's to generalized functions with applications in nonlinear differential equations and control.

Dragomir, S.S. [10] (2014) "Some Inequalities for Trigonometric and Hyperbolic Functions and Their Applications in Optimization" exploring bounds on trigonometric and hyperbolic functions and their direct implications in control theory and optimization.

Bagul, Y.J. and Panchal, S.K.[11] (2018) "Certain Inequalities of Kober and Lazarević Type" RGMIA Research Report Collection, discusses new bounds related to cosine and hyperbolic cosine functions refining Kober-type inequalities, with implications for stability and optimization.

Ames et al., [12] (2019), Control Barrier Functions: Theory and Applications" a comprehensive survey on inequality-driven techniques in control via barrier and Lyapunov functions.

III. EXPLORATION OF INEQUALITIES IN OPTIMIZATION AND CONTROL APPLICATION

In this section present the **e**xploration of inequalities in optimization and control application. The many mathematicians showed various refinement on various inequalities. Especially, in this paper we focused on kober type inequalities that refine classical estimates involving functions like sinx, cosx, tanx, sinhx, coshx, tanhx. These inequalities generally take the form:

$$f(x) \lesseqgtr g(x), \quad \text{for } x \in I \subset R$$

where, f and g are analytic functions. Kober contributed to establishing sharp bounds that are tighter than classical inequalities (e.g., Jordan, Cusa–Huygens, Wilker). Kober extended and refined known inequalities (equations 1 and 2) by applying Taylor expansions with remainder analysis, Convexity or monotonicity properties, Integral mean value theorems, Kober fractional integrals. To apply Kober's inequalities, several mathematical techniques are involved.

Kober's inequalities serve as foundational tools in optimization theory and control system analysis, where bounding nonlinearities is essential for stability, performance, and convergence. Kober-type inequalities can be expressed as sharp bounds or double inequalities for functions such as $\frac{sinx}{x}$, $\frac{cosx}{x}$, $\frac{\tan x}{x}$ and related expressions. Below *i*nequalities hold true and their mathematical implications.

$$\cos\left(\frac{x}{2}\right) < \frac{\sin x}{x} < \frac{2+\cos x}{3} \quad for\ 0 < x < \pi \quad \text{-----------(3)}$$

This classical inequality refines earlier results by Jordan and Cusa. It bounds $\frac{\sin x}{x}$, which frequently appears in control theory (e.g., in sampled-data systems and phase-angle analysis).

$$\cosh\left(\frac{x}{2}\right) < \frac{sinhx}{x} < \frac{2+\cosh x}{3} \quad for\ x > 0 \quad \text{-----------(4)}$$

This hyperbolic analogue is crucial in systems involving exponential growth, damping, or modeling with hyperbolic PDEs. These inequalities are often derived using (a) *Taylor Series Expansions*: For small x, one can expand:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots \quad \text{-----------(5)} \quad ,$$

$$\cos\left(\frac{x}{2}\right) = 1 - \frac{x^2}{8} + \frac{x^4}{384} - \cdots \text{(6)}$$

Comparison of these expansions yields inequalities for different ranges of x.

(b) *Monotonicity and Convexity*: Using the fact that $\boldsymbol{f(x)} = \boldsymbol{\frac{\sin x}{x}}$ is strictly decreasing on (0,π), and convex on suitable intervals, one can establish: $\boldsymbol{x \in (0, \pi)}$, $\boldsymbol{\frac{\sin x}{x}} < 1$

In optimization problems frequently involve nonlinear constraints or cost functions with trigonometric components, for example:

$$\min x \in [0, \pi] \left(\frac{sinx}{x} + \tan x\right), \text{ subject to } \cos x \geq \alpha \quad \text{-----------(7)}$$

Applying equation 3 (Kober-type bounds), that allows to convexify constraints, enabling solution by Lagrange multipliers or interior-point methods. In control system, stability requires that estimating by Lyapunov function:

$$\mathbf{V}^{\cdot}(\mathbf{x}) = -\boldsymbol{a}(\boldsymbol{x})\frac{\boldsymbol{sinx}}{\boldsymbol{x}} \text{ -----------(8)}$$

Using Kober's inequality: $V^{\cdot}(x) < -a(x)\cos\left(\frac{x}{2}\right)$ -----------(9)

provides a tighter decay rate and aids in proving global asymptotic stability. In time delay system, state variables may appear as: x(t−τ) , and sin (x(t−τ)), Inequalities like: $\boldsymbol{sin\,x < x - \frac{x^3}{6}}$ **,** are applied in deriving stability conditions for delayed differential equations. Kober's fractional integrals use in fractional order control, which generalizes classical PID controllers.

$$(K_\alpha f)(x) = \frac{1}{\tau(\alpha)}\int_0^X (x-t)^{\alpha-1} f(t)dt \text{ -----------(10)}$$

IV. RESULTS AND DISCUSSION

Kober-type inequalities is tighter than classical Jordan's inequality, leading to improved precision in system analysis. Because it lead to stronger Lyapunov stability conditions i.e. it offer constructive bounds for Lyapunov functions and their derivatives.

Mathematically, if $V(x) = \frac{1}{2}x^2$ *, and* $V^{\cdot}(x) = x\frac{sinx}{x}$, applying: $V^{\cdot}(x) < -x\cos\left(\frac{x}{2}\right)$

that proves stronger decay, thus better performance.

Kober inequalities facilitation of convex approximations in optimization. i.e In many engineering problems, trigonometric and hyperbolic constraints are non-convex that to **c**onvert non-convex regions to convex approximations**.** This inequality also improved convergence and tractability in energy minimization, optimal path planning and phase control.

Kober's methodology inspired a sequence of related inequalities, forming a hierarchy or chains of bounds with increasing sharpness. i.e establishment of general inequalities like: $\frac{\sin x}{x} < \frac{2+\cos x}{3} < \frac{1+cosx}{2}$ as well as allowing multi-level bounding strategies in control and estimation.

Kober inequalities are used to approximate functions in delayed systems, particularly in Neutral delay differential equations, Predictor-based control systems and Fractional-delay models. Mathematically, expressions like sin (x(t−τ)) can be approximated and bounded using, $\sin x < x - \frac{x^3}{6}$ with error bound estimation. Improving predictive control designs and system compensation.

Controllers designed using Kober-type bounds often show better transient response, lower control energy consumption and greater robustness to disturbances. These are quantified through Lyapunov function values over time and improved integral performance indices like $J = \int_0^\infty (sinx)^2\,dt$ with bounded estimation via Kober.

Kober fractional integrals, $K_\alpha^\mu f(x) = \frac{x^{-\mu-\alpha}}{\tau(\alpha)}\int_0^X t^\mu (x-t)^{\alpha-1} f(t)dt$ used in fractional-order control systems, where the operators allow modeling memory-dependent systems and viscoelastic dynamics, enabling new types of control laws and long-term prediction models.

V. CONSOLIDATED SUMMARY

Kober's trigonometric and hyperbolic inequalities exemplify the power of mathematical abstraction in addressing real-world engineering challenges across diverse application domains. From a theoretical standpoint, Kober-type inequalities allow for the refinement and generalization of classical inequalities by offering tighter bounds and improved convergence behavior. This directly enhances the mathematical rigor of stability analysis, particularly through Lyapunov-based techniques, and facilitates more accurate performance estimation in control systems.

- *Role in Optimization*: These inequalities help transform nonlinear, transcendental constraints into forms more amenable to convex optimization and numerical solvers.
- *Role in Control*: They assist in deriving tighter stability margins, especially in systems with oscillatory or delay characteristics.
- *Mathematical Importance*: Kober-type inequalities often improve or refine classical inequalities (e.g., Jordan, Wilker), leading to sharper analysis results.

VI. CONCLUDING REMARKS

In the context of optimization, these inequalities enable the transformation of non-convex constraints into solvable convex approximations, thereby broadening the scope of tractable problems.

Kober inequalities bridge the gap between non-convex functional expressions and convex optimization techniques, making them vital tools in control-oriented design optimization. Furthermore, their applications extend seamlessly into areas involving time delays, fractional-order dynamics, and systems governed by memory effects, thus making them highly adaptable tools in modern control theory.

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Conflict of Interest: No conflicts of interest.

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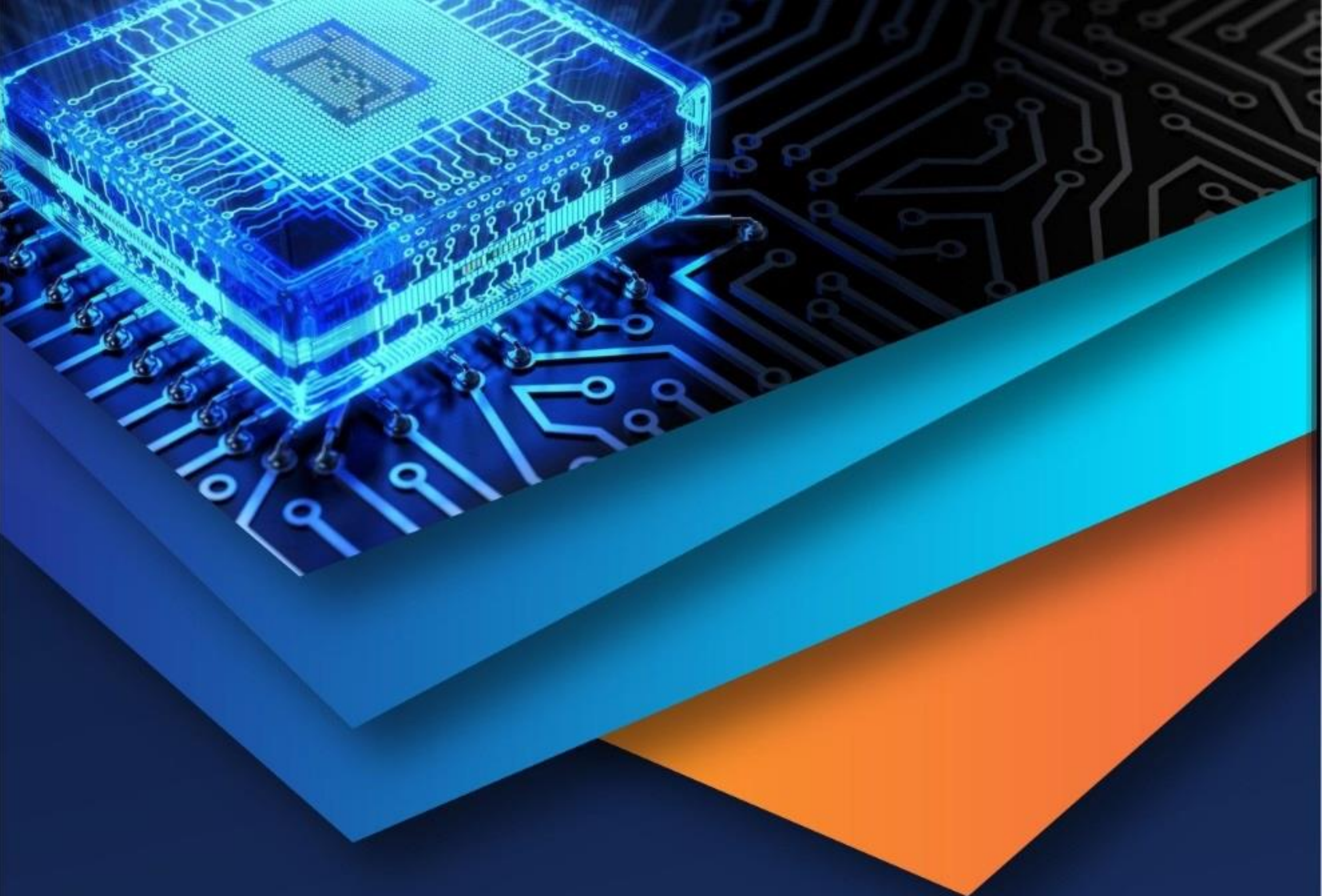

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