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A Structural Study of Morera's Theorem in Multicomplex Spaces

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Abstract: ***In classical complex analysis, Morera's Theorem provides a fundamental characterization of analytic functions via vanishing contour integrals. This note extends the theorem to the setting of multicomplex numbers, which generalize bicomplex and higher-dimensional complex systems. We establish conditions under which a multicomplex-valued function is analytic by examining the vanishing of integrals over suitable closed paths in the multicomplex domain. The result demonstrates that key properties of holomorphic functions persist in the multicomplex framework, thereby enriching the function theory in higher-dimensional commutative algebras.***

AMS Subject Classification (2020)- 30G35,30A05,30E20,30D15.
Keywords and Phrases- Multicomplex analysis; Morera's theorem; analytic functions; contour integrals

I. INTRODUCTION

The theory of bicomplex numbers is a matter of active research for quite a long time since the seminal work of Serge [11] in search of special algebra. The algebra of bicomplex numbers are widely used in the literature as it becomes a viable commutative alternative [12] to the non-commutative skew field of quaternions introduced by Hamilton [6] (both are four-dimensional and generalization of complex numbers) A bicomplex number is defined as

$$\xi = \alpha_0 + i_1\alpha_1 + i_2\alpha_2 + i_1 i_2\alpha_3,$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are real numbers with

$$i_1^2 = i_2^2 = -1$$

and

$$i_1 i_2 = i_2 i_1.$$

The set of bicomplex numbers, complex numbers and real numbers are denoted by $\mathbb{C}_2, \mathbb{C}_1$ and $\mathbb{C}_0$ respectively. $\mathbb{C}_2$ becomes a real commutative algebra with identity

$$1 = 1 + i_1.0 + i_2.0 + i_1 i_2.0$$

with standard binary composition.
There are two non trivial elements

$$e_1 = \frac{1 + i_1 i_2}{2}$$

and

$$e_2 = \frac{1 - i_1 i_2}{2}$$

in $\mathbb{C}_2$ with the properties
$e_1^2 = e_1$,
$e_2^2 = e_2$,

$$e_1 e_2 = e_2 e_1 = 0$$

and

$$e_1 + e_2 = 0$$

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which means that e_1 and e_2 are idempotents alternatively called orthogonal idempotents. By the help of the idempotent elements e_1 and e_2 any bicomplex number $\xi = \alpha_0 + i_1\alpha_1 + i_2\alpha_2 + i_1i_2\alpha_3 = (\alpha_0 + i_1\alpha_1) + i_2(\alpha_2 + i_1\alpha_3) = z_1 + i_2z_2$, where $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}_0$,can be expressed as $\xi = z_1 + i_2z_2 = \xi_1e_1 + \xi_2e_2$ where and

$$\xi_1 = z_1 - i_1z_2$$

and

$$\xi_2 = z_1 + i_1z_2 \in \mathbb{C}_1$$

This representation of a bicomplex number is known as the idempotent representation of ξ, ξ_1, ξ_2 and are called the idempotent components of the bicomplex number $\xi = z_1 + i_2z_2$, resulting a pair of mutually complementary projections:

$$P_1 : (z_1 + i_2z_2) \in \mathbb{C}_2 \to (z_1 - i_1z_1) \in \mathbb{C}_1$$
$$P_2 : (z_1 + i_2z_2) \in \mathbb{C}_2 \to (z_1 + i_1z_1) \in \mathbb{C}_1.$$

An element $\xi = z_1 + i_2z_2$ is singular if and only if $|z_1^2 + z_2^2| = 0$.

The set of singular elements is denoted as O_2 and is defined by

$O_2 = \{\xi \in \mathbb{C}_2 : \xi \text{ is the collection of all complex multiple of } e_1 \text{ and } e_2\}$.

The norm $\|.\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ (the set of all non negative real numbers) of a bicomplex number is defined as $\|\xi\| = \sqrt{|z_1^2| + |z_2^2|} = \sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2}$.

The bicomplex algebra can be generalized to the multicomplex algebras $\mathbb{C}_n$, defined inductively by introducing n mutually commuting imaginary units $i_1, i_2, \ldots i_n$ with $i_k = -1$. Elements of $\mathbb{MC}_n$ are of the form

$$z = \sum_{k=1}^{n} z_k i_k,$$

where $z_k \in \mathbb{C}_1$ and i_k ranges over products of the imaginary units. Similar to the bicomplex case, multicomplex algebras possess idempotent decompositions that allow reduction to several classical complex components. The idempotent structure becomes increasingly intricate as n grows, but the analytic principles remain parallel.

For further study on multicomplex analysis one can see [3] and [4].

In $\mathbb{C}_1$ we may now state the following theorem due to Morera:

Theorem A: If f is continuous in a simply connected domain D and if $\int_\Gamma f(z)dz = 0$ for every closed rectifiable curve Γ in D then f is analytic in D.

II. MAIN THEOREMS

Theorem 2.1. Let Ω be a domain in $\mathbb{MC}_n$ and $f : \mathbb{MC}_n \to \Omega$ be a continuous function. If for every closed rectifiable curve $\Gamma : [0,1] \to \Omega$ with $\Gamma(0) = \Gamma(1)$,

$$\int_\Gamma f(\zeta)d\zeta = 0$$

holds then f is holomorphic on Ω.

Proof. We choose arbitrarily a fixed point z_0 in Ω.

Let $\gamma: [0,1] \to \Omega$ be a piecewise continuous curve such that

$$\gamma(0) = z_0$$

and

$$\gamma(1) = z_0$$

where z_0 is any point in Ω.

We define a function

$$F: \Omega \to \mathbb{MC}_n \text{ by } F(z) = \int_\Gamma f(\zeta)d\zeta .$$

We have to show that

(i) F is well defined and

(ii) $F'(z_0)$ exists for z_0 running over Ω.

Let Φ^{-1} be another piecewise continuous curve in Ω with opposite orientation that of γ.

So,

$$\Phi^{-1}(t) = \gamma(1 - t) \text{ , for } 0 \le t \le 1.$$

Then clearly,

$$\Phi^{-1}(0) = z_0 \text{ and } \Phi^{-1}(1) = z.$$

We know that for every $\in \mathbb{MC}_n$, it has a representation

$$z = \sum_{k=1}^{2^{n-1}} z_k e_k.$$

Let γ_k be the projection of γ on the z_k −plane whose starting point is z_k^0 and end point is z_k for all $k = 1,2,\ldots,2^{n-1}$.
Since γ is a rectifiable curve, γ_k's are also rectifiable curves in z_k −plane for all $k = 1,2,\ldots,2^{n-1}$ {cf.[2]}. Now since f is a multicomplex valued function, it has decomposition

$$f(z) = \sum_{k=1}^{2^{n-1}} f_k(z_k)e_k \text{ for } z = \sum_{k=1}^{2^{n-1}} z_k e_k.$$

So,

$$F(z) = \int_\Gamma f(\zeta)d\zeta = \sum_{k=1}^{2^{n-1}} \left(\int_{\gamma_k} f_k(\zeta_k)d\zeta_k\right) e_k.$$

The curve ψ consisting of γ followed by Φ^{-1} is a piecewise continuous closed curve in Ω. Therefore ψ has a projection $\gamma_k \circ \Phi_k^{-1}$ in z_k −plane is also piecewise continuous closed curve.
Hence the integral of f over the curve ψ is zero. Thus,

$$0 = \int_\psi f(\zeta)d\zeta = \int_\gamma f(\zeta)d\zeta + \int_{\Phi^{-1}} f(\zeta)d\zeta$$
$$= \int_\gamma f(\zeta)d\zeta - \int_\Phi f(\zeta)d\zeta$$

So,

$$\int_\gamma f(\zeta)d\zeta = \int_\Phi f(\zeta)d\zeta \qquad (1)$$

Also,

$$\int_{\gamma_k \circ \Phi_k^{-1}} f_k(\zeta_k)d\zeta_k = 0 \Longrightarrow \int_{\gamma_k} f_k(\zeta_k)d\zeta_k = \int_{\Phi_k^{-1}} f_k(\zeta_k)d\zeta_k, \forall k = 1,2,\ldots,2^{n-1} \qquad (2)$$

Therefore from (1), (2), we say that

$$F(z) = \int_\Gamma f(\zeta)d\zeta$$

is well-defined.
We choose a point $h \in \mathbb{MC}_n$ is so small that $z_0 + h \in \Omega$.
Moreover, we choose a continuous curve μ: $[0,1] \to \Omega$ such that $\Phi(0) = z_0$ and $\Phi(1) = z_0 + h$, i.e.,Φ is a curve joining the points z_0 and $z_0 + h$.
Now,

$$f(z_0) = \frac{f(z_0)}{h}\int_\Phi d\zeta, \text{ where } h \text{ is non-singular.}$$
$$= \frac{1}{h}\int_\Phi f(z_0)d\zeta$$

So,

$$\left\|\frac{F(z_0 + h) - F(z_0)}{h} - f(z_0)\right\|$$
$$= \left\|\frac{1}{h}\left(\int_\Phi f(\zeta)d\zeta - f(z_0)\right)\right\|$$
$$= \left\|\frac{1}{h}\left(\int_\Phi [f(\zeta) - f(z_0)]\,d\zeta\right)\right\|$$

Since f is continuous at z_0, so for every $\epsilon > 0$, there exists a $\delta > 0$, such that when ever,

$$\|(z_0 + h) - z_0\| < \delta, \|f(\zeta) - f(z_0)\| < \frac{\epsilon\|h\|}{2L}.$$

Now,

$$\left\|\frac{1}{h}\left(\int_\Phi [f(\zeta) - f(z_0)]\, d\zeta\right)\right\|$$
$$\leq \sqrt{2}\frac{1}{\|h\|}\left\|\left(\int_\Phi [f(\zeta) - f(z_0)]\, d\zeta\right)\right\|$$
$$< \sqrt{2}\frac{1}{\|h\|}.\sqrt{2}\frac{\epsilon\|h\|}{2L}\int_\Phi \|d\zeta\|$$
$$= \sqrt{2}\frac{1}{\|h\|}.\sqrt{2}\frac{\epsilon\|h\|}{2L}.L = \epsilon$$

where L is the length of Φ, which shows that,

$$\lim_{h\to 0}\frac{\mathrm{F}(z_0+\mathrm{h})-\mathrm{F}(z_0)}{\mathrm{h}} = f(z_0),$$

h being non-singular i.e., $F'(z_0) = f(z_0)$.
Therefore F is holomorphic on Ω. Since F is holomorphic on Ω, F′ is also holomorphic on Ω, which implies that f is holomorphic on Ω.

This proves the theorem.
Remark 2.1. In connection to the functions of single complex variable, it is well known that Morera's Theorem can be regarded as a sort of converse of Cauchy's Fundamental Theorem. In view of Theorem 2.1 Cauchy's Fundamental Theorem may be deduced under multicomplex treatment and therefore this can be though of as an open problem in this connection. Thus naturally all the consequences of Cauchy's Fundamental Theorem valid for functions of single complex variable can also be treated similarly.

III. APPLICATION IN ARTIFICIAL INTELLIGENCE

Multicomplex versions of Morera's Theorem and Cauchy's Fundamental Theorem may matter for AI because they give us better math for high-dimensional problems. Fields like image analysis, quantum-inspired computing, neural networks, and signal transforms all deal with data that isn't just 2D or 3D. Multicomplex analytic functions provide a richer algebraic and geometric language to describe that data.
This broader notion of analyticity could lead to new feature extraction methods, better multidimensional optimization, and more stable training. It's especially relevant for complex-valued neural nets, quantum AI, and data representations where multiple imaginary components interact. So bringing Morera's Theorem into multicomplex analysis isn't only a theoretical step — it might drive new tools for AI.

IV. ACKNOWLEDGMENT

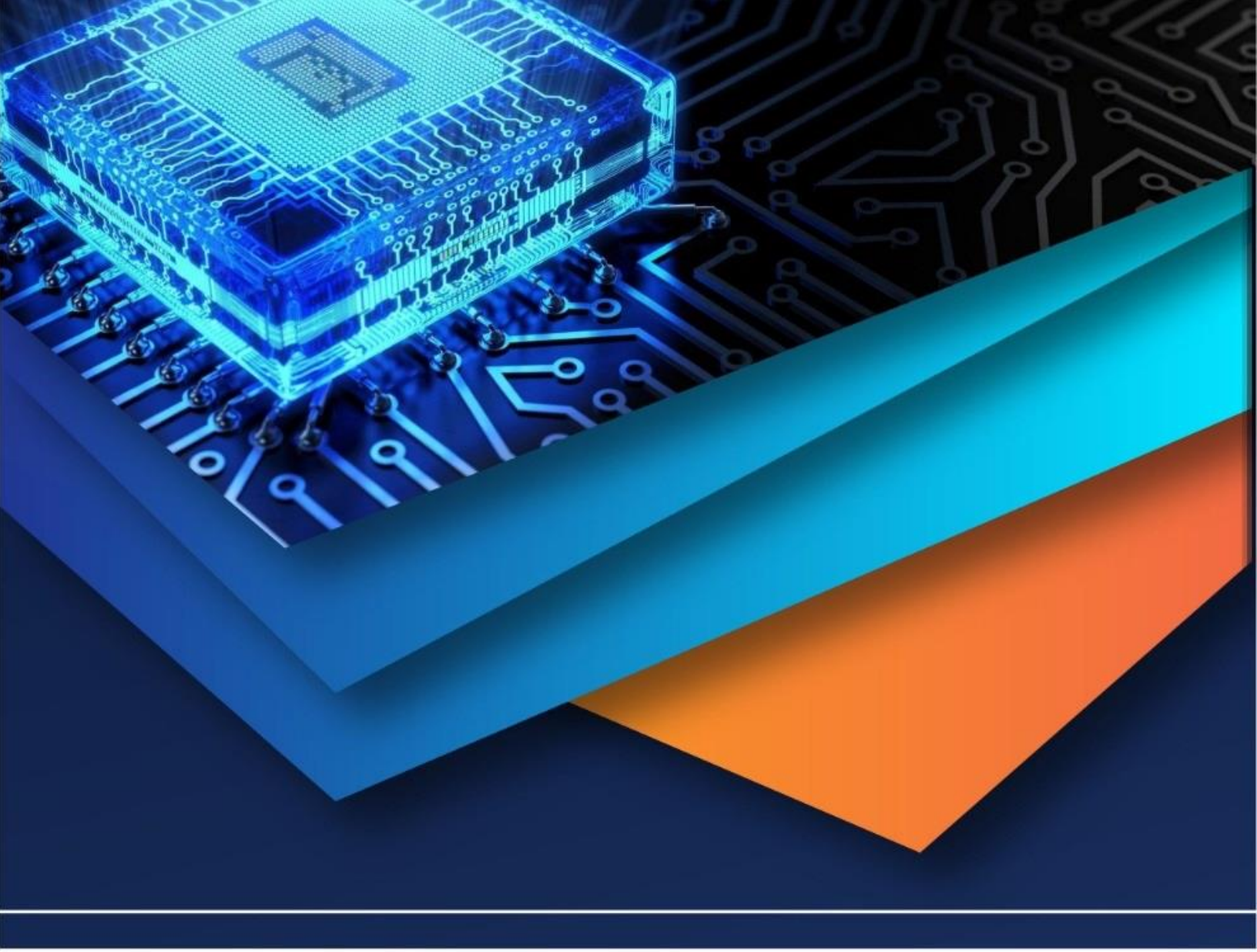
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