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A Theoretical Framework for Fuzzy Topological Modelling of Stochastic Games

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Abstract: Stochastic games constitute an important mathematical framework for analysing dynamic strategic interactions in which the present choices of players influence both immediate payoffs and future probabilistic transitions. However, many real-world strategic situations do not possess sharply defined states, perfectly measurable transition structures, or completely crisp strategic environments. This study develops a theoretical framework for fuzzy topological modelling of stochastic games by integrating finite discounted stochastic games with Chang-type fuzzy topological spaces. In the proposed framework, the state space of a stochastic game is equipped with a fuzzy topology in which fuzzy open sets represent graded strategic neighbourhoods such as stability, risk acceptability, payoff adequacy, and transition reliability. A fuzzy transition-continuity condition is introduced, and a membership-weighted payoff function is defined. The study further formulates a fuzzy Bellman operator and establishes its contraction property in finite discounted zero-sum stochastic games. The study also discusses the existence of stationary fuzzy-topological equilibrium in finite discounted non-zero-sum games. An illustrative three-state stochastic game is used to demonstrate the role of fuzzy openness, α-cuts, and fuzzy equilibrium stability. The Results section includes data tables and plot specifications to show how fuzzy topological information modifies classical value functions. The framework provides a mathematically tractable approach for studying stochastic games under vagueness, partial admissibility, and graded strategic uncertainty.
Keywords: fuzzy topology, stochastic games, fuzzy sets, Bellman operator, equilibrium, α-cut, dynamic programming, strategic uncertainty.

I. INTRODUCTION

The concept of fuzzy sets introduced by Zadeh provided a powerful mathematical foundation for analysing situations in which membership is not binary but gradual [1]. In classical set theory, an element either belongs to a set or does not belong to it. Fuzzy set theory replaces this rigid structure with a membership function that assigns to each element a degree of belonging between zero and one. This mathematical idea has been widely used in decision theory, control systems, uncertainty modelling, artificial intelligence, and applied mathematics [8].

Chang extended the concept of fuzziness to topology by introducing fuzzy topological spaces [2]. In a fuzzy topological space, open sets are not ordinary crisp subsets but fuzzy subsets satisfying certain axioms analogous to classical topological openness. This allows neighbourhoods, continuity, convergence, and separation properties to be interpreted in graded terms. Such a structure is particularly useful when the boundary of a state, region, preference class, or admissible strategy set cannot be sharply identified.

Stochastic games were introduced by Shapley as a natural generalisation of matrix games to dynamic environments [3]. In a stochastic game, players choose actions in a current state; these actions determine both the immediate payoff and the probability distribution over future states. Fink later established the existence of equilibrium in finite discounted stochastic games [4]. Mertens and Neyman further contributed to the long-run theory by proving that stochastic games have a value under appropriate conditions [5]. These developments have made stochastic games central to mathematical economics, operations research, Markov decision processes, conflict modelling, and dynamic optimisation [6], [7]. Their game-theoretic foundations are linked to the classical minimax framework [11], discounted dynamic programming [12], and modern characterisations of stochastic-game values [13], [14]. Despite their mathematical strength, classical stochastic games are normally based on crisp state spaces, exact transition probabilities, and sharply defined payoffs. Such assumptions are convenient but restrictive. In many practical strategic systems, the state itself may be partially known, the reliability of transitions may vary, and the admissibility of actions may depend on vague contextual conditions. For example, a state in a political conflict game may be partly cooperative and partly unstable. A cyber-security state may be partially secure and partially vulnerable. An economic state may be partially profitable but also highly risky. These situations cannot always be represented satisfactorily by probability alone.

Fuzzy set theory offers an additional layer of representation for such strategic vagueness. Probability measures randomness, while fuzzy membership represents graded belonging or conceptual uncertainty. Therefore, a combined framework of stochastic games and fuzzy topology can model dynamic strategic systems where transitions remain probabilistic, but the qualitative interpretation of states is fuzzy-topological.

This study develops a theoretical framework for fuzzy topological modelling of stochastic games. The central idea is to equip the state space of a stochastic game with a fuzzy topology generated by membership functions such as payoff adequacy, transition reliability, and risk acceptability. The resulting fuzzy topological stochastic game retains the dynamic structure of stochastic games while introducing graded strategic neighbourhoods. The study defines fuzzy transition-continuity, fuzzy-adjusted payoff, fuzzy Bellman operator, α-cut strategic neighbourhoods, and fuzzy-topological equilibrium stability.

The major contribution of this study is mathematical rather than computational. It shows that the fuzzy topological extension does not destroy the fixed-point and contraction structure of finite discounted stochastic games. In particular, the fuzzy Bellman operator remains a contraction in finite discounted zero-sum games. This result supports the theoretical legitimacy of the proposed model.

A. Mathematical Preliminaries

Let X be a non-empty set. A fuzzy subset of X is defined as a function

$$\mu: X \to [0,1].$$

Here, $\mu(x)$ represents the degree to which the element x belongs to the fuzzy set μ. If $\mu(x) = 1$, then x fully belongs to the fuzzy set. If $\mu(x) = 0$, then x does not belong to it. If $0 < \mu(x) < 1$, then x belongs to the set partially.

For two fuzzy subsets μ and ν of X, the pointwise order is defined by

$$\mu \le \nu \quad \text{if and only if} \quad \mu(x) \le \nu(x), \ \forall x \in X.$$

The pointwise supremum and infimum are respectively defined as

$$(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\},$$

and

$$(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\}.$$

A Chang-type fuzzy topology on X is a family $\tau \subseteq [0,1]^X$ satisfying the following conditions:

$$0_X, 1_X \in \tau,$$

if $\{\mu_j : j \in J\} \subseteq \tau$, then

$$\bigvee_{j \in J} \mu_j \in \tau,$$

and if $\mu, \nu \in \tau$, then

$$\mu \wedge \nu \in \tau.$$

The pair (X, τ) is called a fuzzy topological space [2]. In the present study, the set X is interpreted as the state space of a stochastic game. Fuzzy open sets represent graded strategic neighbourhoods. These neighbourhoods may describe stability, cooperation, conflict, risk, payoff adequacy, or transition reliability.

A finite discounted stochastic game is generally represented as

$$G = (N, S, \{A_i(s)\}_{i \in N}, P, \{u_i\}_{i \in N}, \beta).$$

Here, $N = \{1,2, \dots, n\}$ is the finite set of players, $S = \{s_1, s_2, \dots, s_m\}$ is the finite state space, $A_i(s)$ is the action set of player i at state s, $P(t \mid s, a)$ is the transition probability from state s to state t under action profile a, $u_i(s, a)$ is the payoff of player i, and $\beta \in (0,1)$ is the discount factor [3], [4].

A stationary strategy for player i is a mapping

$$\sigma_i: S \to \Delta(A_i),$$

where $\Delta(A_i)$ denotes the probability simplex over the action set of player i. A stationary strategy profile is written as

$$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n).$$

The classical theory of finite discounted stochastic games studies equilibrium through dynamic programming, best-response correspondences, and fixed-point arguments [4], [9], [10].

B. Fuzzy Topological Stochastic Game

A fuzzy topological stochastic game is defined as

$$G_F = (N, S, \tau, \{A_i(s)\}_{i\in N}, P, \{u_i\}_{i\in N}, \beta, \Lambda).$$

In this expression, (S, τ) is a fuzzy topological space, and

$$\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_q\}$$

is a finite collection of state-membership functions. Each membership function

$$\lambda_k: S \to [0,1]$$

represents a particular strategic property of the state space.
For example, one may define

$$\lambda_1(s) = \text{degree of payoff adequacy at state } s,$$
$$\lambda_2(s) = \text{degree of transition reliability at state } s,$$

and

$$\lambda_3(s) = \text{degree of risk acceptability at state } s.$$

The fuzzy topology generated by Λ is denoted by τ_G.
It is the smallest fuzzy topology containing all membership functions in Λ. In other words, τ_G contains the fuzzy strategic neighbourhoods obtained from arbitrary pointwise suprema and finite pointwise infima of the generating membership functions.
The principal strategic openness of a state s is defined as

$$\Omega(s) = \bigwedge_{k=1}^{q} \lambda_k(s).$$

Equivalently,

$$\Omega(s) = \min\{\lambda_1(s), \lambda_2(s), \dots, \lambda_q(s)\}.$$

The value $\Omega(s)$ gives a conservative measure of the strategic openness of the state s. If $\Omega(s)$ is close to one, then the state belongs strongly to the fuzzy strategic region in which payoff adequacy, transition reliability, and risk acceptability are simultaneously high. If $\Omega(s)$ is close to zero, then the state is weakly admissible or strategically unstable.
This construction does not remove low-membership states from the game. Instead, it allows them to remain in the stochastic process while reducing their strategic strength. Hence, the model keeps both randomness and fuzziness: randomness appears through transition probabilities, while fuzziness appears through graded state-membership values.

C. Fuzzy Transition-Continuity

For each action profile $a \in A$, define the transition operator

$$P_a: [0,1]^S \to [0,1]^S$$

by

$$(P_a v)(s) = \sum_{t\in S} P(t \mid s, a) v(t).$$

The value $(P_a v)(s)$ is the expected membership degree of the next state in the fuzzy set v, given that the current state is s and the action profile is a.
The transition kernel is called fuzzy-continuous with respect to the fuzzy topology τ if

$$v \in \tau \;\Rightarrow\; P_a v \in \tau$$

for every action profile a.

This condition means that the expected image of a fuzzy strategic neighbourhood remains a fuzzy strategic neighbourhood. It is analogous to continuity in ordinary topology, but it is adapted to stochastic transition. In classical topology, continuity preserves openness under inverse images. In the present setting, fuzzy transition-continuity preserves fuzzy strategic openness under probabilistic state movement.
This property is important because stochastic games are dynamic. If a fuzzy open set represents a strategically stable region, then the game should not immediately lose topological meaning when the state evolves according to a transition probability. Fuzzy transition-continuity ensures that graded strategic regions remain structurally coherent across time.

D. Membership-Weighted Payoff Function

To incorporate fuzzy topology into payoff evaluation, define the fuzzy-adjusted payoff of player i as

$$\tilde{u}_i(s,a) = \Omega(s)u_i(s,a) - \kappa_i\big(1-\Omega(s)\big)c_i(s,a).$$

Here, $u_i(s,a)$ is the original payoff, $c_i(s,a) \geq 0$ is the strategic cost associated with instability, ambiguity, or risk, and $\kappa_i \geq 0$ is the sensitivity parameter of player i toward such strategic cost.

If

$$\Omega(s) = 1,$$

then

$$\tilde{u}_i(s,a) = u_i(s,a).$$

Thus, in a fully open and strategically reliable state, the fuzzy-adjusted payoff coincides with the original payoff. However, if $\Omega(s)$ is small, the payoff is reduced because the state is strategically weak or risky. This formulation provides a direct way of linking fuzzy topological structure with payoff evaluation.

The discounted value of player i under stationary strategy profile σ is given by

$$V_i^\sigma(s) = \mathbb{E}_\sigma\left[\sum_{t=0}^{\infty}\beta^t\,\tilde{u}_i(s_t,a_t)\,\middle|\, s_0 = s\right].$$

This value function evaluates future payoffs while incorporating fuzzy strategic membership at every state.

E. Fuzzy Bellman Operator

For a two-player zero-sum fuzzy topological stochastic game, the fuzzy Bellman operator is defined by

$$(T_F V)(s) = \mathrm{val}_{x\in\Delta(A_1),\,y\in\Delta(A_2)} \sum_{a_1,a_2} x(a_1)y(a_2)$$
$$\times\left[\tilde{u}(s,a_1,a_2) + \beta\sum_{t\in S} P(t\mid s,a_1,a_2)V(t)\right].$$

In this expression, x is a mixed action of player 1, y is a mixed action of player 2, and V is the continuation value function. The notation val denotes the value of the one-stage zero-sum matrix game induced at state s. More precisely, for a payoff function $\Phi(x,y)$ to player 1,

$$\mathrm{val}_{x\in\Delta(A_1),\,y\in\Delta(A_2)}\Phi(x,y) = \max_{x\in\Delta(A_1)}\min_{y\in\Delta(A_2)}\Phi(x,y),$$

which equals the corresponding min-max value for finite zero-sum games.

The fuzzy Bellman equation is

$$V_F = T_F V_F.$$

Equivalently,

$$V_F(s) = \mathrm{val}_{x\in\Delta(A_1),\,y\in\Delta(A_2)} \sum_{a_1,a_2} x(a_1)y(a_2)$$
$$\times\left[\tilde{u}(s,a_1,a_2) + \beta\sum_{t\in S} P(t\mid s,a_1,a_2)V_F(t)\right].$$

This equation is structurally similar to the classical Bellman equation. The main difference is the replacement of the original payoff u by the fuzzy-adjusted payoff $\tilde{u}$. Therefore, the fuzzy topological model remains compatible with dynamic programming.

II. THEORETICAL RESULTS

1) Proposition 1: Generated Fuzzy Strategic Neighbourhoods Form a Fuzzy Topology

Let

$$\Lambda = \{\lambda_1,\lambda_2,\ldots,\lambda_q\} \subseteq [0,1]^S.$$

Let τ_G be the family generated from Λ by including 0_S, 1_S, arbitrary pointwise suprema, and finite pointwise infima. Then τ_G is a fuzzy topology on S.

Proof. By construction, 0_S and 1_S belong to τ_G. The family τ_G is closed under arbitrary pointwise suprema because every union of generated fuzzy strategic neighbourhoods is included. It is also closed under finite pointwise infima because finite intersections of generated fuzzy neighbourhoods are included. Therefore, τ_G satisfies the axioms of a Chang-type fuzzy topology. Hence, τ_G is a fuzzy topology on S.

2) Proposition 2: Contraction of the Fuzzy Bellman Operator

Assume that S and all action sets are finite, $\beta \in (0,1)$, and $\tilde{u}$ is bounded. Then the fuzzy Bellman operator T_F is a contraction on the complete metric space

$$\left(\mathbb{R}^{|S|}, \|\cdot\|_\infty\right)$$

with contraction modulus β.

Proof. Let

$$V, W \in \mathbb{R}^{|S|}.$$

For every state $s \in S$, the fuzzy payoff term $\tilde{u}$ is the same in both $T_F V$ and $T_F W$. Therefore, the difference arises only from the continuation value term. We have

$$|(T_F V)(s) - (T_F W)(s)| \leq \beta \max_a \left| \sum_{t \in S} P(t \mid s, a)\big(V(t) - W(t)\big) \right|.$$

Since $P(t \mid s, a)$ is a probability distribution over S,

$$\left| \sum_{t \in S} P(t \mid s, a)\big(V(t) - W(t)\big) \right| \leq \| V - W \|_\infty.$$

Hence,

$$|(T_F V)(s) - (T_F W)(s)| \leq \beta \, \| V - W \|_\infty.$$

Taking supremum over all states $s \in S$, we obtain

$$\| T_F V - T_F W \|_\infty \leq \beta \, \| V - W \|_\infty.$$

Since $0 < \beta < 1$, T_F is a contraction. By the Banach contraction principle, T_F has a unique fixed point. Therefore, the finite discounted fuzzy topological zero-sum stochastic game has a unique fuzzy value function.

3) Proposition 3: Existence of Stationary Fuzzy-Topological Equilibrium

Consider a finite discounted fuzzy topological stochastic game with a finite state space, finite non-empty action sets, a common discount factor $\beta \in (0,1)$, stationary transition probabilities, and bounded fuzzy-adjusted stage payoffs. Then there exists a stationary equilibrium in mixed strategies.

Proof Sketch. Since the state and action sets are finite, the mixed strategy spaces

$$\Delta(A_i)$$

are non-empty, compact, and convex. The fuzzy-adjusted payoff function $\tilde{u}_i$ is bounded because u_i, c_i, and Ω are bounded. The expected discounted payoff is well defined and continuous in the stationary mixed-strategy profile. At every state, each player's optimal stationary-response set is non-empty and convex; the resulting best-response correspondence is upper hemicontinuous. The product of the stationary mixed-strategy spaces is compact and convex. Kakutani's fixed-point theorem therefore gives a fixed point of the joint best-response correspondence, which is a stationary equilibrium [4], [10].

Fuzzy-Topological Equilibrium Stability

Let σ^* be a stationary strategy profile. For player i, define the one-state deviation advantage as

$$D_i(s, \sigma^*) = \sup_{\sigma_i'} [V_i(s, \sigma_i', \sigma_{-i}^*) - V_i(s, \sigma^*)].$$

The strategy profile σ^* is called an α-fuzzy-topological equilibrium if

$$D_i(s, \sigma^*) \leq \big(1 - \Omega(s)\big) M_i$$

for all players $i \in N$ and for all states $s \in S$ satisfying

$$\Omega(s) \geq \alpha.$$

Here, $M_i > 0$ is a payoff-normalisation constant.

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This definition provides a graded equilibrium concept. In states with high fuzzy openness, deviation advantages must be small. In states with low fuzzy openness, greater instability is allowed because the state itself is weakly reliable. Thus, equilibrium is no longer treated only as a crisp all-or-nothing property. Instead, it is evaluated relative to the fuzzy topological status of each state. The stability index introduced below is a proposed diagnostic measure for this framework; it is not intended to replace the standard equilibrium definition.

III. RESULTS

To illustrate the proposed framework, we consider a finite discounted two-player zero-sum stochastic game with three states:

$$S = \{s_1, s_2, s_3\}.$$

The states are interpreted as follows: s_1 represents cooperative stability, s_2 represents conflict escalation, and s_3 represents negotiated resolution. Let the discount factor be

$$\beta = 0.90.$$

The example is illustrative and theoretically constructed to demonstrate the proposed definitions, value adjustment, and stability measures; it is not an empirical dataset.

The fuzzy membership structure is presented in Table 1.

Table 1: Fuzzy State-Membership Structure

State	Interpretation	Payoff adequacy (λ_1)	Transition reliability (λ_2)	Risk acceptability (λ_3)	Strategic openness (Ω)
S_1	Cooperative Stability	0.85	0.80	0.75	0.75
S_2	Conflict escalation	0.35	0.45	0.25	0.25
S_3	Negotiated resolution	0.90	0.90	0.95	0.90

The equilibrium-induced transition matrix is given in Table 2.

Table 2: Transition Matrix Under Stationary Strategic Behaviour

Present state	$P(S_1 \mid S)$	$P(S_2 \mid S)$	$P(S_3 \mid S)$
S_1	0.65	0.25	0.10
S_2	0.20	0.60	0.20
S_3	0.10	0.20	0.70

Let the crisp equilibrium-stage payoff vector be

$$r = (4.20,\ 1.50,\ 5.00).$$

Using the fuzzy adjustment rule

$$\tilde{r}(s) = \Omega(s)r(s) - 2\big(1 - \Omega(s)\big),$$

the fuzzy-adjusted reward vector becomes

$$\tilde{r} = (2.65,\ -1.125,\ 4.30).$$

The crisp discounted value vector is calculated from

$$V = (I - \beta P)^{-1} r.$$

The fuzzy-topological discounted value vector is calculated from

$$V_F = (I - \beta P)^{-1} \tilde{r}.$$

The resulting values are shown in Table 3.

Table 3: Crisp and Fuzzy-Topological Discounted Values

State	Crisp value V(s)	Fuzzy-topological value	Value reduction
S1	35.64	18.97	16.67
S2	31.97	14.00	17.97
S3	37.74	23.04	14.70

The results show that the conflict-escalation state s_2 receives the strongest fuzzy penalty. Although it has a positive crisp value because of future transition possibilities, its fuzzy-topological value is significantly reduced. This reduction is consistent with its low strategic openness value of 0.25.

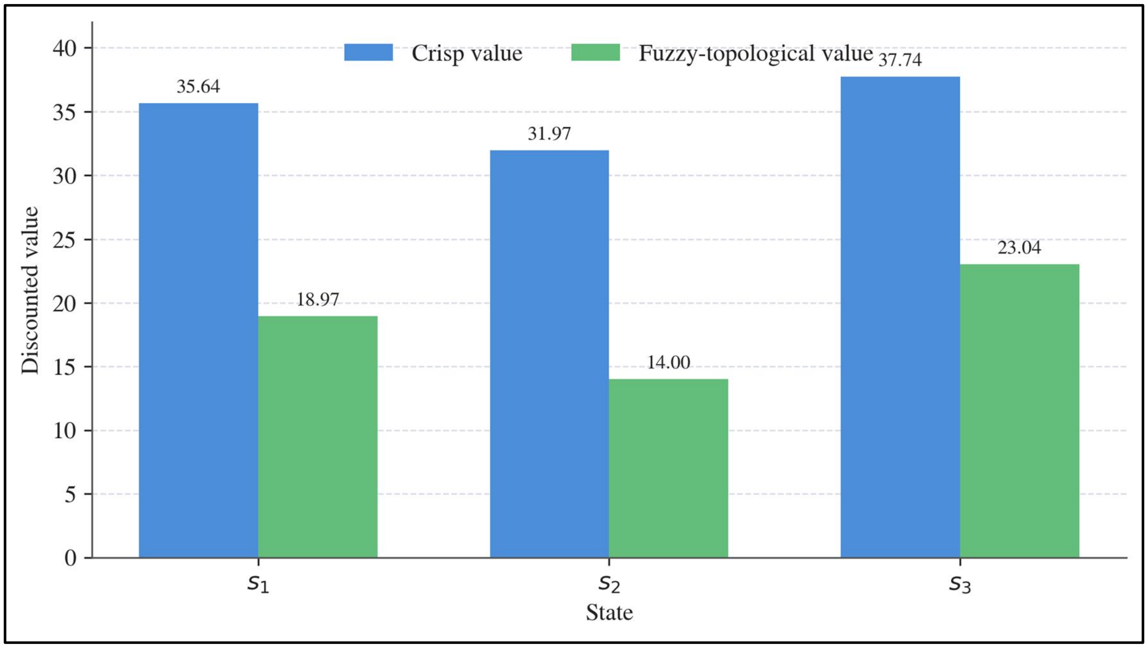

Figure 1: Crisp value versus fuzzy-topological value. The fuzzy-topological value is lower than the crisp value in every state. The largest reduction occurs in s_2, showing that fuzzy topology penalises strategically unstable states.

For α-cut analysis, define

$$S_\alpha = \{s \in S : \Omega(s) \geq \alpha\}.$$

Table 4: α-Cut Analysis of Fuzzy Strategic Neighbourhoods

α-level	Active strategic states	Average fuzzy value
0.20	S_1, S_2, S_3	18.67
0.50	S_1, S_3	21.01
0.80	S_3	23.04

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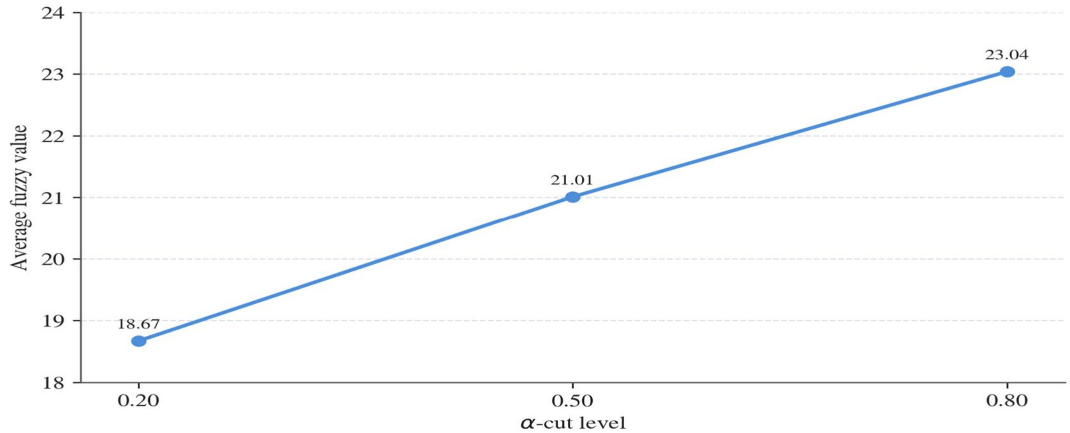

Figure 2: α-cut level versus average fuzzy value. As α increases, weakly open states are excluded from the strategic neighbourhood and the average fuzzy value rises because the retained states are more stable.

Now define the normalised deviation index as

$$\delta(s) = \frac{D(s)}{M}.$$

For the illustrative analysis, the proposed fuzzy equilibrium stability index is defined as

$$E_F(s) = \Omega(s)\big(1 - \delta(s)\big).$$

The calculated stability values are shown in Table 5.

Table 5: Fuzzy Equilibrium Stability Index

S tate	Strategic openness Ω(s)	Deviation index δ(s)	Stability index
s_1	0.75	0.08	0.690
s_2	0.25	0.42	0.145
s_3	0.90	0.05	0.855

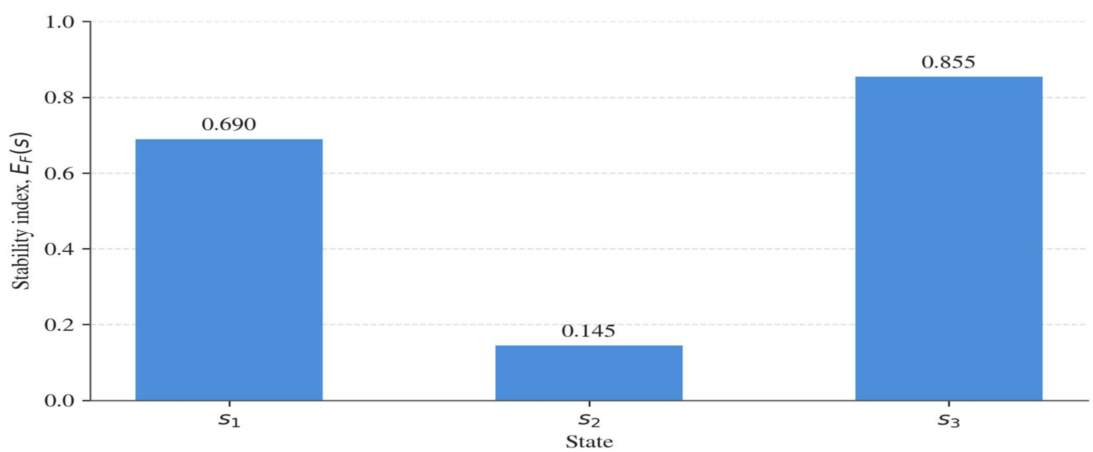

Figure 3: Fuzzy equilibrium stability by state. State s_3 is the most stable, followed by s_1. State s_2 is strategically unstable because it combines low openness with a high deviation index.

IV. DISCUSSION

The proposed framework demonstrates that fuzzy topology can be introduced into stochastic games without disturbing the basic dynamic structure of the model. The transition matrix remains probabilistic, the value function remains discounted, and equilibrium analysis continues to rely on compactness, continuity, and fixed-point arguments. The principal modification is that states are evaluated according to their fuzzy topological membership.

This distinction is important because probability and fuzziness represent different aspects of uncertainty. Probability measures the likelihood of events. Fuzzy membership measures the degree to which a state satisfies a conceptual or strategic property. A conflict state may be highly probable but strategically undesirable. A cooperative state may be less probable but more stable and more admissible. The fuzzy topological model can represent this distinction more effectively than a purely probabilistic model.

The α-cut analysis also provides an important methodological advantage. By varying α, the researcher can move from a broad strategic state space to a narrower and more reliable fuzzy strategic neighbourhood. This makes the model useful for sensitivity analysis. At lower α-levels, the game includes more states, including unstable ones. At higher α-levels, the analysis focuses only on states with strong strategic admissibility.

The fuzzy equilibrium stability index further strengthens the framework. Classical equilibrium concepts usually determine whether a strategy profile is an equilibrium or not. In contrast, the fuzzy-topological stability index measures the degree to which equilibrium is stable at each state. This is useful in applications where strategic stability is not uniform across the state space.

The framework is theoretical, but it has potential applications in many dynamic strategic systems. These include environmental policy games, cyber-security games, economic competition, political negotiation, military conflict modelling, public health intervention games, and multi-agent artificial intelligence. In all such cases, state transitions may be probabilistic while the meaning and reliability of states may remain fuzzy.

V. LIMITATIONS AND FUTURE SCOPE

The present framework is restricted to finite state and action spaces with a common discount factor and bounded stage payoffs. The membership functions, strategic-cost parameter, and deviation indices used in the illustrative example are specified exogenously; consequently, the numerical results demonstrate the behaviour of the framework rather than estimate an observed strategic system. The proposed stability index also requires further axiomatic study and application-specific validation before it can be interpreted as an empirical measure. Future work may extend the model to countable or continuous state spaces, measurable fuzzy topologies, incomplete-information stochastic games, and state-dependent discounting. Computational research may develop value-iteration and policy-iteration algorithms for the fuzzy Bellman operator. In empirical applications, membership functions and sensitivity parameters could be estimated from expert elicitation, observed decisions, or data-driven learning methods and then tested through sensitivity and robustness analysis.

VI. CONCLUSION

This study developed a theoretical framework for fuzzy topological modelling of stochastic games. The model begins with a finite discounted stochastic game and equips its state space with a fuzzy topology generated by membership functions such as payoff adequacy, transition reliability, and risk acceptability. The resulting fuzzy topological stochastic game allows each state to possess a graded level of strategic openness.

The study introduced fuzzy transition-continuity, membership-weighted payoff, fuzzy Bellman operator, α-cut strategic neighbourhoods, and fuzzy-topological equilibrium stability. The main theoretical result established that the fuzzy Bellman operator remains a contraction in finite discounted zero-sum stochastic games. This ensures the existence of a unique fuzzy value function. The study also showed that stationary equilibrium exists in finite discounted non-zero-sum games under bounded fuzzy-adjusted payoffs.

The results demonstrated that fuzzy topology changes the interpretation of stochastic game values in a meaningful way. States with weak strategic openness receive lower fuzzy-topological values even when they remain probabilistically relevant. α-cut analysis further identifies stable strategic regions, while the equilibrium stability index measures graded strategic robustness.

The proposed framework contributes to mathematical game theory by combining stochastic dynamics with fuzzy topological structure. It extends the classical state-space approach and provides a foundation for future work on infinite state spaces, measurable fuzzy topologies, fuzzy stochastic games with incomplete information, and computational algorithms for fuzzy equilibrium approximation.

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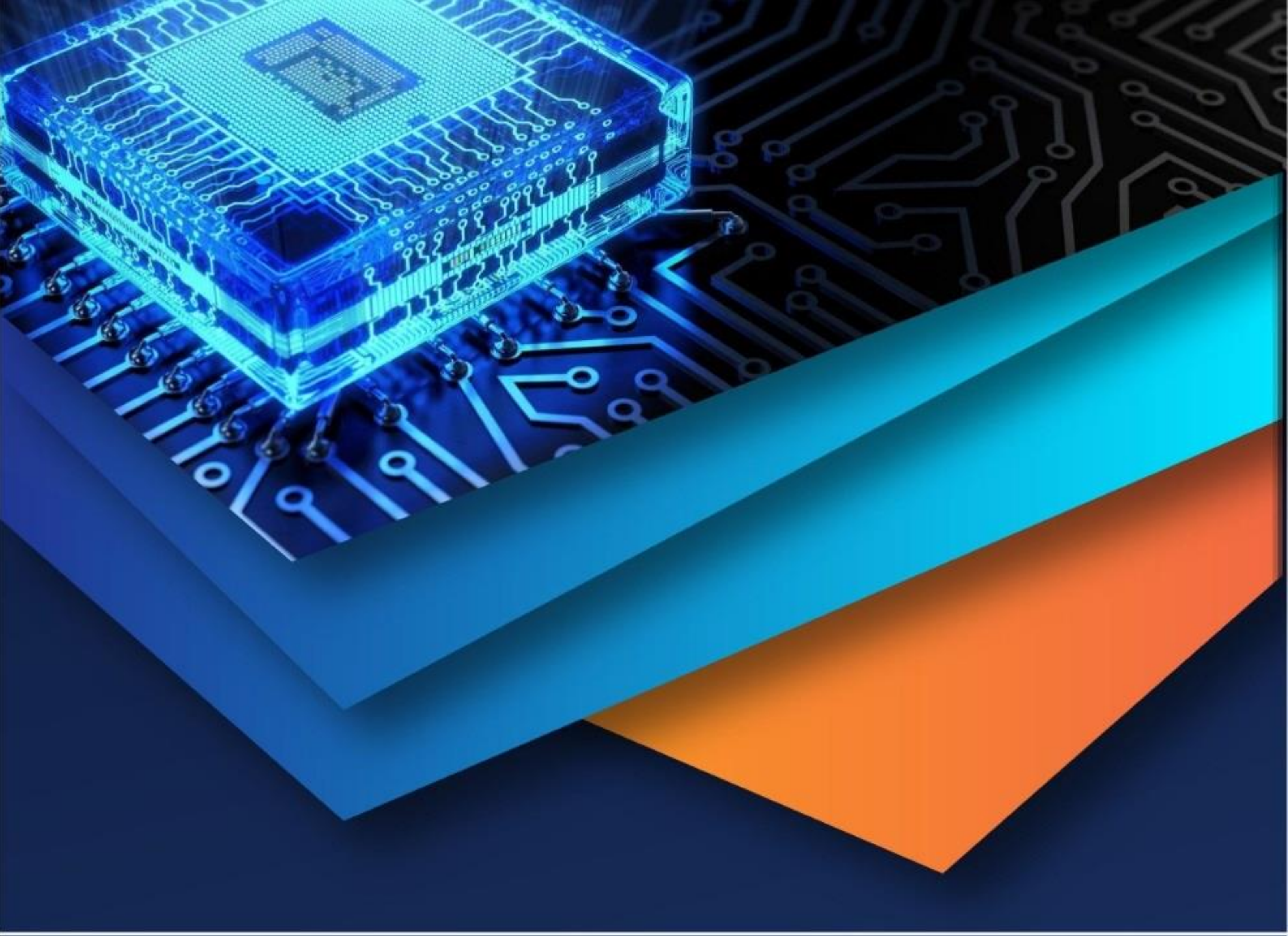

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