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An Algorithm for Computing the n^{th} Root of Real Numbers using a General Method

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Abstract: ***Based on an old article about extracting the n^{th} of real numbers using a general method, we introduce an algorithm for computing such roots, using a different approach, through introducing an operator which we define as the inverse binomial operator .We illustrate the procedure by computing n^{th} roots of some ,randomly, selected real numbers.***
To show the beauty of using such a new procedure, we compare our calculations with the calculations obtained, for the a certain example, via well-known iteration methods such as the bisection and Newton-Raphson methods; moreover a comparison is made for same purpose using continued fractions method. Finally, a conclusion is given to clarify the advantages of the inverse binomial operator.
Keywords: General method, inverse binomial operator, root, real numbers, bisection.

I. INTRODUCTION

Computing roots of real numbers has been a problem of interest since the early days of mathematics. The square root is usually calculated by hand at the preparatory level, in the Libyan schools, with different ways and one of these methods is so amusing, inspiring and fruitful; the method was generalized to all n^{th} roots of real numbers [1].
In this paper we treat the problem in a more sophisticated manner through the use of the inverse binomial operator(IBO) and make computations for some examples supported with comparison of this general procedure with well-known iterations from the literature.
In section II we revise some of the concepts related to the methods of computing $n^{th}\ roots\ of\ real\ numbers\ x(\in R), such\ that\ n \geq 2$; then in section III ,we introduce the general procedure using the IBO, and then we illustrate our procedure by giving full calculations for some examples, and in section IV ,we compare the results obtained for Example 5 (in section III) with calculations using other iteration methods and where we chose here the Newton-Raphson(NRM) and bisection(BM) methods, along with the method of continued fractions(CNM) [2].Finally, in section V ,we conclude with a discussion which reveals the cause behind using this general method, stressing its beauty ,its easiness and exactness .

II. ELEMENTARY CONCEPTS

We begin with revising a method for getting the square root of real numbers; we show that through an example:

A. *Example 1*

To compute $\sqrt{1225}$,using our inspiring method.
Method of solution

1) *Step 1:* We form the table below ,where we house in each square two digits starting with units of the number in question.
2) *Step 2:* Find the largest number such that its square is less than or equal to 12 ,which is 3 here .Subtract the square of this number (9) from 12 and form the number composed of the remainder of this subtraction and the next two digits to get 325.Now we have obtained the first digit from root which is 3 and this digit is put in the left top-most square.
3) *Step 3:* The obtained digit 3 is then multiplied by 2 $[=\binom{2}{1}], which\ is\ the\ binomial\ coefficient)$ leading to the result 6; this is put in the tens position; this number is shown in the table below to the left. We then search for the largest number(which is 5 here) such that the product of this number and the number formed of that in the brackets and this same number(=65),is less than or equal to the number obtained in Step 2 (i.e. 325). We subtract and continue the same process followed in Step 3 if necessary .Hence the result in this case is given by $\sqrt{1225} = 35$ as shown in Table 1.[again the digit 5 form the next digit of the root as shown in Table 1.]

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Table 1.the evaluation of $\sqrt{1225}$.

$\sqrt{1225}$ =	3	5
3 × 3	12 -9	25
(6)5 × 5	3 -3	25 25
	00	00

Note that the number we are searching its root is a whole square and is an integer; however we can also find roots for numbers with decimal digits in the same manner putting in mind that after arranging the integer part as mentioned above, we put the decimal part following the digital part two by two as exemplified below

3	25	17.	45	00	43	12

B. Example 2

To find the square root $\sqrt{15625}$,we refer to Table 2 which is self- explanatory.

Table 2.the evaluation of $\sqrt{15625}$.

$\sqrt{15625}$ =	1	2	5
1 × 1	1 -1	56	25
	0	56	
(2)2 × 2		-44	
(24)5 5		12 -12	25 25
		00	00

Therefore $\sqrt{15625} = 125$.

We now proceed to rely on this inspiring method, to generalize to the cube root of real numbers; where we need here to use the two binomial coefficients $\binom{3}{1} = 3$ and $\binom{3}{2}$=3. Moreover, we put three digits in each square starting with the unit digit ,and for the first digit of the root we search for the largest number whose cube is less than or equal to the number(a say), in the first square ,of our given number we need to calculate its cube root .while for the rest of the digits of the root we follow the same procedure we have shown when solving for the square root, but now with the following modifications:

The number between brackets in the position of hundreds is $[\binom{3}{2} a^2]$, and the number in the tens position is $[\binom{3}{1} a]$,finally the number in the units position is a; and if the new digit of the root is guessed as b ,then the number to be subtracted is

$$10^2(3a^2)b + 10(3a)b^2 + b^3 \quad (1)$$

We illustrate this in the following example

C. Example 3

Using the general method to compute $\sqrt[3]{9261}$.

Table 3.Evaluation of $\sqrt[3]{9261}$ by a general method [1].

$\sqrt[3]{9261}$ =	2	1
2^3	9 -8	261
$(3 \times 2^2)(3 \times 2)1$ 1 1	1 -1	261 261
	0	000

III. INVERSE BINOMIAL OPERATOR

Now to generalize for the n^{th} root for a real-valued number x , we begin with forming the table with n digits in each square beginning with the unit digit, and as shown in the previous examples if the determined number so far is a , b is the proper guess for the next digit of the root; and if we consider the different binomial coefficients

$$p_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} , r = 1,2, \dots , n-1 \qquad (2)$$

Then the inverse binomial operator, which acts on , is defined as

$$O(b) = p_{n-1}(10a)^{n-1}b + p_{n-2}(10a)^{n-2}b^2 + \cdots + p_1(10a)^1 b^{n-1} + b^n \qquad (3)$$

By applying our operator we actually calculated the quantity $Q = O(b)$ to be subtracted.
Note that the very first digit in the table has its special treatment as noted in the above examples .Moreover the operation in Equation(3) can be better explained by the scheme drawn below[Figure(1)] [1].

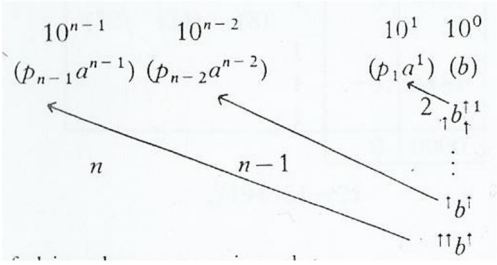


Figure (1)-Schematic drawing for the operation in Equation(3) [1].

Now we introduce our algorithm using the general procedure for extracting the n^{th} root for a real-valued number ;

1) We form the appropriate table and house n digits in each square starting from the digit of the units and if the number has decimal digits then they are housed in the same manner to the right of the decimal point.
2) The first digit of the root is the largest number(a say) such a^n is less than or equal to the part of the number under consideration x in the first square(to the left)(n_1 say) ;and we evaluate $n_1 - a^n \equiv r_1$.The digit a is now the first digit(from left) of the required root.
3) The number r_1 is combined appropriately with r_2 which is the part of x in the second square; and with a in mind we search for a number b which fulfills Equation(3) ; once we get b and make the subtraction ,we obtain the second digit of the root.
4) We continue applying step 3 until we reach the required accuracy ,noting that all obtained digits of the root are exact. Moreover, the required guesses do not exceed two.

According to the algorithm mentioned above, we give the following example

A. *Example 4*

Using the above algorithm ,evaluate $\sqrt[4]{194481}$.
Solution
Since $p_1 = 4$, $p_2 = 6$. $p_3 = 4$,and using the above algorithm, we get $\sqrt[4]{194481} = 21$, as shown in Table 4.

Table 4.Evaluation of $\sqrt[4]{194481}$ using the algorithm [1].

$\sqrt[4]{194481}$ =	2	1
2^4	19 -16	4481
$(4 \times 2^3)(6 \times 2^2)(4 \times 2)$1 1 1 1	3 -3	4481 4481
	0	0000

B. Example 5

Compute $\sqrt[3]{56423456}$ using the algorithm [3]

Solution

On the lights of our algorithm ,the solution is given in Table 5, and where the result is given up to five decimal figures.

Table 5.The value of $\sqrt[3]{56423456}$ up to five decimal figures.

$\sqrt[3]{56423456}$=	3	8	3.	5	4	8	1	5
3^3	56 -27	423	456.	000	000	000	000	000
{3(3^2)}{3(3)} 8 8 8	29 -27	423 872						
{3(38^2)}{3(38)} 3 3 3	1 -1	551 309	456 887					
{3(383^2)}{3(383)}5 5 5	0	241 -220	589 320	000 875				
{3(3835^2)}{3(3835)} 4 4 4		21 -17	288 650	125 510	000 864			
{3(38354^2) {3(38354)} 8 8 8		3 -3	617 530	614 543	136 998	000 592		
{3(383548^2)}{3(383548)}1 1 1		0	87 -44	070 132	137 731	408 997	000 641	
{3(3835481^2)}{3(3835481)}5 5 5			42 -22	937 006	405 374	410 628	359 652	000 375
			20	871	030	781	706	625

C. Remarks

1) Except for the first digit, the procedure to be followed is as shown in the scheme in Figure(1) , which is based on the operation in Equation(3)(IBO),

2) In the case of a rational number ,the decimal part is housed with the decimal point as the start; as shown in Example 5.

3) The initial guess b can easily be obtained by looking at the number in the table (to be subtracted from) and the number $p_{n-1}a^{n-1}$.

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4) The method is a descending one in the sense that the digits determined first are the highest in order . e.g . if the root is 21 as in example 4 ,then the digit 2 is the one to be determined first.

5) Using the above mentioned procedure, one can easily carry on the calculation by hand .Patience is needed when dealing with large numbers especially if they are rational ones [4].

IV. THE USE OF SOME ITERATION METHODS

Here ,we present two methods which are well known in the literature:

A. Newton-Raphson Method (NRM)

The problem is formulated as follows

" Solve the problem $x^n - c = 0$,where $n \geq 2$; $c\ is\ a\ real\ number$."

This is equivalent to the statement "Find the n^{th} root of the real number c". To use Newton-Raphson method(NRM),we need to calculate the derivative of $f(x) = x^n - c = 0$; which is given by $f'(x) = nx^{n-1}$. The recurrence for getting the root is then

$$x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m)}\ , m = 0,1,2,..\ and\ f'(x) \neq 0 \qquad (4)$$

This NRM which is supposed to be the most quick convergent method .Of course we need to assume an initial guess , e.g. an appropriate initial guess for Example 5 is 380[5].For our case of extracting roots of real numbers, we see that the recurrence in Equation(4), becomes

$$x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m)} = x_m - \frac{x_m^n - c}{nx_m^{n-1}}\ , m = 0,1,2,.. \qquad (5)$$

The accuracy needed is achieved by using a certain tolerance value ϵ ;namely, the calculations ended when

$$|x_{m+1} - x_m| < \epsilon \qquad (6)$$

Where ϵ is usually taken as something like 10^{-6}(and 10^{-10} if necessary) to get an accuracy up to five decimal figures [5].

A program is written for this purpose using different versions of matlab(2010,2016,9.4) ,where the initial guess is taken as 300 and the tolerance as 10^{-10}.The required root with exact five decimal figures is obtained after six iterations [3]The value of the root in this case is found to be 3.8354815197e+002.

B. Bisection method

Another famous method to solve algebraic equation is the bisection method(BM). In this method and if $f(x) = 0$, we search for two points $x_1\ and\ x_2\ such\ that\ f(x_1) and\ f(x_2)\ are\ with\ different\ signs$. we calculate the midpoint of the two points , i.e. $x_3 = (x_1 + x_2)/2$ and check the sign of $f(x_3)$; and if its sign is the same as $f(x_1)\ (say)$, then we replace x_1 by x_3 , and calculate $x_4 = (x_3 + x_2)/2$, and again check the sign of $f(x_4)$ repeating the same process until the two points are squeezed to the required value of the root[see Figure (2) [5].

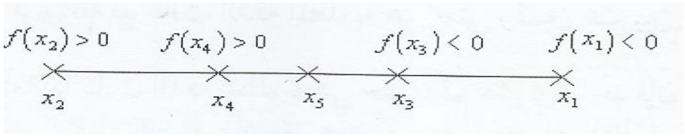


Figure (2). The bisection method[5].

Note that our function here is $f(x) = x^n - c = 0$,and a tolerance value is needed to stop the calculation [5]. One should also note that this method is a little bit lengthy and tedious ,although it is still being used [6].For Example 5, the BM gave the same answer with five significant figures ;but now 17 iterations were needed to get that result.

Now before summarizing the results obtained from the NRM and BM in solving the equation $f(x) = x^n - c = 0$ for Example 5 as a sample , we defer the discussion of these results in comparison with IBM till we present another method depending on the use of continued fractions.

C. Continued Fraction Method (CFM)

At the beginning we give few definitions related to the subject of continued fractions(CF), then we pass directly to their use in solving $f(x) = x^n - c = 0$; moreover, we give the results for $\sqrt[3]{56423456}$ in Example 5 (given when studying the IBM).

1) Definition 1

An expression of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots}}} \tag{7}$$

Is called a continued fraction; and where a_i , $i = 1,2,3,..$ and $b_i, i = 1,2,3,..$ are real numbers.

If it happens that the numerator in each fraction is equal to one, then the continued fraction is a simple one. i.e. $a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}$ is

a simple continued fraction [1].

Note that any normal fraction($\frac{p}{q}$, $p \text{ and } q \text{ are integers and } q \neq 0$) can be expressed as a continued fraction as [1]

$$\frac{p}{q} = a + \cfrac{1}{b + \cfrac{1}{c + \ddots}} \equiv [a; b, c, \dots] \tag{8}$$

2) Definition 2

The finite continued fraction in a simple one with a finite number of continued ones. i.e. it has the form

$$[a_0; a_1, a_2, \dots\dots, a_n] \tag{9}$$

3) Definition 3

An infinite continued fraction is expressed in terms of an infinite number of continued ones; namely it has the form[2] [3]

$$[a_0; a_1, a_2, \dots\dots] \tag{10}$$

Now with these definitions in mind ,we turn to their applications to get the $n^{th}\ root\ of\ real\ numbers$.

Here, we will deal with the n^{th} root of real numbers such that $n \geq 2$.We write $r = (z)^{\frac{1}{n}} = (a^n + b)^{\frac{1}{n}} = a + c; z\ is\ a\ real\ number$;then $a^n + b = (a + c)^n$.Using the binomial theorem ,we get [6]

$$a^n + b = a^n + na^{n-1}c + \frac{n(n-1)a^{n-2}c^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}c^3}{3!} + \cdots \tag{15}$$

With straightforward simplifications we get

$$c = \cfrac{b}{na^{n-1} + \cfrac{n(n-1)a^{n-2}c}{2!} + \cfrac{n(n-1)(n-2)a^{n-3}c^2}{3!} + \cdots} \tag{16}$$

Hence our first guess c_1 is

$$c_1 = \frac{b}{na^{n-1}} \tag{17}$$

And the first approximation for the root is

$$r_1 = a + c_1 = a + \frac{b}{na^{n-1}} \tag{18}$$

Using Equation(16) and Equation(17) ,we get the second guess in c as

$$c_2 = \cfrac{b}{na^{n-1} + \cfrac{b(n-1)}{2a}} \tag{19}$$

The second approximation for r is then

$$r_2 = a + c_2 = a + \cfrac{b}{na^{n-1} + \cfrac{b(n-1)}{2a}} \tag{20}$$

In general we get

$$r = (z)^{\frac{1}{n}} = (a^n + b)^{\frac{1}{n}} = a + \cfrac{b}{na^{n-1} + \cfrac{b(n-1)}{2a + \cfrac{b(n+1)}{3na^{n-1} + \cfrac{b(2n-1)}{2a + \cfrac{b(2n+1)}{5na^{n-1} + \cdots}}}}} \tag{21}$$

Using Equation(21) and for Example(5), we see that $n = 3, a = 383, b = 241569$. Moreover, $\sqrt[3]{56423456}$ can be written as

$$\sqrt[3]{56423456} = 383 + \cfrac{241569}{440067 + \cfrac{241569}{383 + \cfrac{483138}{1320201 + \cfrac{1207845}{766 + \cdots}}}}$$

Hence the first approximation for the required root is $r_1 = 383$,from which we see that the second approximation r_2 is $r_2 = 383 + \frac{241569}{440067} = 383.5489$. The third approximation is given by $r_3 = 383 + \frac{241569}{440067+\frac{241569}{383}} = 3838.548$; we continue in this manner to get r_5 as $r_5 = 383 + \frac{241569}{440067+\frac{241569}{383+\frac{483138}{1320201+\frac{1207845}{766}}}} = 383.5481519'$,which is the required root (precise up to five decimal points). This result is magnificently consistent with the result obtained for the same problem using IBM(≡ IBO).

In Table 6 ,we present a comparison between the four different methods for the sample example(Example 5);where the root was to be $\sqrt[3]{56423456} = 383.54815$ O((five significant decimal figures).

Table 6.Comaprison of the various method.

Method	Number of steps or iterations	Remarks
NRM	6	Quick convergence
BM	17	Slow convergence
CFM	5	Very good convergence
IBM	8	So precise

The table shows that the three methods :NMR, CFM, and IBM are comparable; while the BM is somewhat slow as expected.

V. CONCLUSION

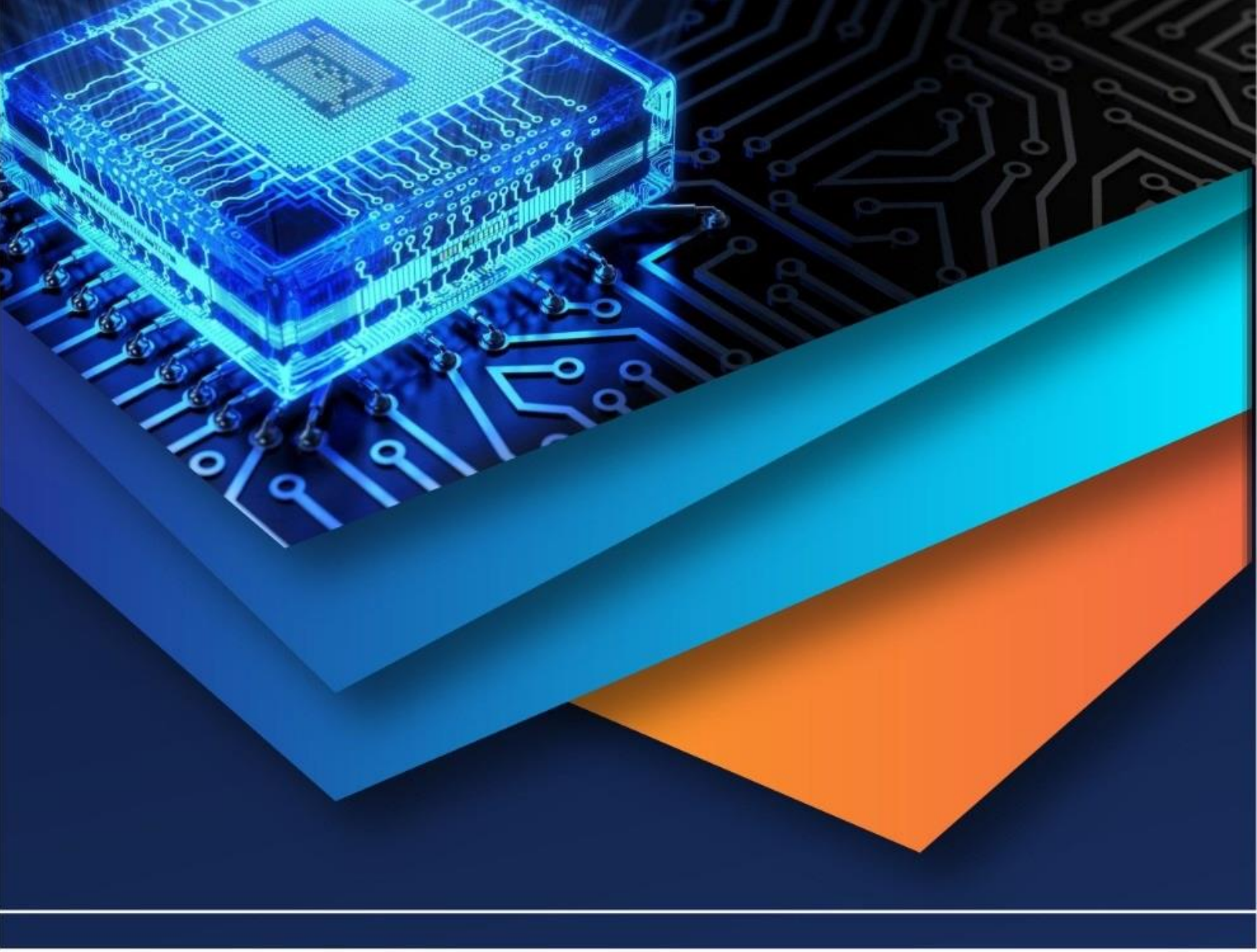
It is shown ,in this work ,that computing the n^{th} roots of real numbers is still of vital interest and importance .e.g. we point to a method dealing with the determination of the n^{th} root of positive real numbers, which is a digit-by-digit method up to any desired accuracy [7]. The IBM is also a digit-by-digit method ,and proved to be comparable with NRM and CFM .Extra to that ,the IBM proved to be so precise ,elegant and fruitful. Moreover ,the method can be more beneficial if the algorithm is supported by the right software. Finally, we note that the real number should be positive when we want to extract its $n^{th}\ root$, $when\ n\ is\ even$.

VI. ACKNOWLEDGEMENT

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