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An Analysis on the Ternary Cubic Diophantine Equation $2(l^2+m^2)-3lm=56t^3$

G. Janaki[1], S. Shanmuga Priya[2]
[1, 2]Department of Mathematics, Cauvery College for Women, Tiruchirappalli, Tamil Nadu, India

Abstract: In this article, we concentrate on identifying all the non-zero, infinitely many integral solutions to the ternary cubic equation $2(l^2+m^2)-3lm=56t^3$. Of these solutions, some exciting patterns are discussed.
Keywords: Diophantine equation, Integral Solutions, Ternary Cubic equation withthree unknowns.

I. INTRODUCTION

The universal language of the world is mathematics, which imparts knowledge of numbers, structures, formulas and shapes. Integers and integral valued functions are studied in the branch of pure mathematics known as Number theory. A polynomial equation with at least two unknowns that has only integer solutions is known as a Diophantine equation. The term "Diophantine" refers to *Diophantus* of Alexandria, a third-century Hellenistic mathematician who studied these equations and was one of the first to introduce symbolismto algebra. Number theory is discussed in [3, 4, 9, 11] whereas in [6] Quadratic Diophantine equation is analysed. In [1, 2, 5, 7, 8, 10], the authors have considered cubic equation for study. In this work, a non homogeneous ternary cubic equation with three unknowns $2(l^2+m^2)-3lm=56t^3$ is considered in order to find some of its interesting integral solutions.

A. Notations

1) $SO_n = n(2n^2-1)$ = Stella Octangula number of rank n
2) $Gno_n = 2n-1$ = Gnomonic number of rank n
3) $RD_n = (2n-1)(2n^2-2n+1)$ = Rhombic dodecagonal number of rank n
4) $Star_n = 6n(n-1)+1$ = Star number of rank n
5) $CC_n = (2n-1)(n^2-n+1)$ = Centered cube number of rank n
6) $TT_n = \frac{1}{6}(23n^2-27n+10)$ = Truncted tetrahedral number of rank n
7) $T_{10,n} = n(4n-3)$ = Decagonal number of rank n
8) $TO_n = 16n^3-33n^2+24n-6$ = Truncated octrahedral number of rank n
9) $T_{17,n} = \frac{n(15n-13)}{2}$ = Heptadecagonal number of rank n
10) $T_{3,n} = \frac{n(n+1)}{2}$ = Triangular number of rank n
11) $P_n^7 = \frac{1}{6}n(n+1)(5n-2)$ = Heptagonal pyramidal number of rank n
12) $P_n^5 = \frac{n^2(n+1)}{2}$ = Pentagonal Pyramidal number of rank n
13) $T_{25,n} = \frac{n(23n-21)}{2}$ = Icosipentagonal number of rank n

II. PROBLEM ANALYSIS

Consider the following ternary cubic Diophantine equation for study.

$$2(l^2+m^2)-3lm=56\,t^3 \tag{1}$$

Take the linear transformations

$$l=r+s \qquad m=r-s \tag{2}$$

Applying (2) in (1) leads to the form

$$r^2+7s^2=56t^3 \tag{3}$$

Assume that

$$t=c^2+7d^2 \tag{4}$$

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A. *Pattern 1*

Write

$$56 = (7 + i\sqrt{7})(7 - i\sqrt{7}) \qquad (5)$$

Using (4) and (5) in (3) results in

$$(r + i\sqrt{7}\,s)(r - i\sqrt{7}s) = (7 + i\sqrt{7})(7 - i\sqrt{7})\,(c + i\sqrt{7}d)^3(c - i\sqrt{7}d)^3$$

Considering the like terms on both sides,

$$(r + i\sqrt{7}s) = (7 + i\sqrt{7})(c + i\sqrt{7}d)^3 \qquad (6)$$

On equating the real and imaginary parts of the above equation, the following equations are obtained.

$$r = 7c^3 - 147cd^2 - 21c^2d + 49d^3 \qquad (7)$$

$$s = c^3 - 21cd^2 + 21c^2d - 49d^3 \qquad (8)$$

With the help of (7), (8) and (2) the integral solution of (1) can be obtained as follows.

$$l = 8c^3 - 168cd^2 \qquad (9)$$

$$m = 6c^3 - 126cd^2 - 42c^2d + 98d^3 \qquad (10)$$

$$t = c^2 + 7d^2 \qquad (11)$$

Properties

1) $l(1,1) + m(1,1) + t(1,1) \equiv 0\ (mod\ 216)$
2) $l(c,c) + m(c,c) + ct(c,c)$ is a cubic number
3) $l(c,1) - 4SO_c + 82Gno_c \equiv 0\ (mod\ 82)$
4) $m(c,1) - RD_c - CC_c + star_c - 6TT_c - T_{10,c} - 84Gno_c - c + 123$ is a nasty number
5) $3t(1,1)$ and $4t(1,1) - m(1,1)$ are nasty numbers.
6) $-10l(1,1)$ is a perfect square.

B. *Pattern 2*

Equation (3) can be viewed as

$$r^2 + 7s^2 = 56t^3.1 \qquad (12)$$

Express '1' as

$$1 = \frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{4^2}$$

(12) leads to

$$(r + i\sqrt{7}\,s)(r - i\sqrt{7}s) = \frac{(7 + i\sqrt{7})(7 - i\sqrt{7})(3 + i\sqrt{7})(3 - i\sqrt{7})(c + i\sqrt{7}d)^3(c - i\sqrt{7}d)^3}{4^2}$$

Equating like terms on both sides,

$$(r + i\sqrt{7}\,s) = \frac{(7 + i\sqrt{7})(3 + i\sqrt{7})(c + i\sqrt{7}d)^3}{4}$$

On comparing the real and imaginary parts of the above equation,

$$r = \frac{1}{4}[14c^3 - 210c^2d - 294cd^2 + 490d^3] \qquad (13)$$

$$s = \frac{1}{4}[10c^3 + 42c^2d - 210cd^2 - 98d^3] \qquad (14)$$

Our aim is to find the integer solution of (1). Hence, Taking $c = 4C$ and $d = 4D$ in (13), (14), (4) and applying the resulting values of r, s in (2), the integral solution of (1) can be found as follows,

$$l = 4^2[24C^3 - 168C^2D - 504CD^2 + 392D^3]$$

$$m = 4^2[4C^3 - 252C^2D - 84CD^2 + 588D^3]$$

$$t = 4^2[C^2 + 7D^2]$$

Properties:

1) $m(1,1)$ and $3m(1,1) - l(1,1)$ are perfect squares.
2) $l(C,C)$ and $m(C,C)$ are cubic numbers.
3) $l(C,1) + m(C,1) - 32[CC_C - TO_C + 24T_{17,C}] + 7536\,Gno_C \equiv 0\ (mod\ 8304)$
4) $m(1,1) - 5t(1,1)$ is a nasty number

C. Pattern 3

Define

$$56 = \frac{(7 + i5\sqrt{7})(7 - i5\sqrt{7})}{2^2}$$

Utilizing the above value in (3),

$$(r + i\sqrt{7}\,s)(r - i\sqrt{7}s) = \frac{(7 + i5\sqrt{7})(7 - i5\sqrt{7})(c + i\sqrt{7}d)^3(c - i\sqrt{7}d)^3}{2^2}$$

Connecting the like terms,

$$(r + i\sqrt{7}\,s) = \frac{(7 + i5\sqrt{7})(c + i\sqrt{7}d)^3}{2}$$

From the above equation, the values of r and s can be obtained by comparing the real and imaginary components on both sides as

$$r = \frac{1}{2}[7c^3 - 147cd^2 - 105c^2d + 245d^3] \quad (15)$$

$$s = \frac{1}{2}[5c^3 - 105cd^2 + 21c^2d - 49d^3] \quad (16)$$

In order to obtain the integral solution of (1), choosing $c = 2C$ and $d = 2D$ in (15),(16) and (4) provides the solution of the form

$$l = 2^3[6C^3 - 126CD^2 - 42C^2D + 98D^3] \quad (17)$$

$$m = 2^3[C^3 - 21CD^2 - 63C^2D + 147D^3] \quad (18)$$

$$t = 2^2[C^2 + 7D^2] \quad (19)$$

With the help of $c = 2C$ and $d = 2D$, the following integral solution of (1) can be obtained.

$$l = 6c^3 - 126cd^2 - 42c^2d + 98d^3$$

$$m = c^3 - 21cd^2 - 63c^2d + 147d^3]$$

$$t = 2^2[c^2 + 7d^2]$$

Properties

1) $l(1,1)$ and $t(1,1)$ are cubic numbers.
2) $m(1,1)$ is both cubic as well as a perfect square number.
3) $m(1,1) - l(1,1) \times t(1,1)$ is a nasty number.
4) $t(c,1) - 2T_{3,c} + c \equiv 0\ (mod\ 7)$
5) $2P_c^5 - m(c,1) - 2T_{25,c} - 82T_{3,c} - c\ t(1,0) \equiv 0\ (\text{mod } 147)$

III. CONCLUSION

In this article, we have made an effort to obtain the integral solution of the non-homogeneous ternary cubic equation. Furthermore, one may search for another pattern of integral solution for the considered equation.

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