International Journal For Research in Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 10 **Issue:** IV **Month of publication:** April 2022

DOI: https://doi.org/10.22214/ijraset.2022.41358

Application of Runge – Kutta Method to Population Equations

Arunachalam Sundaram
Department of Mathematics, Rizvi College of Arts, Science and Commerce, University of Mumbai, Mumbai, Maharashtra, India.

***Abstract:** In this paper, we implement the second order Runge – Kutta method for three different population initial value problems. The Runge – Kutta method is a numerical technique used to solve the approximate solution for initial value problems for ordinary differential equations. Runge – Kutta method is implemented to linear population equation, non-linear population equation and non-linear population equation with an oscillation. The method of solving three initial value problems is implemented using Python Programming.*
***Keywords:** Differential equation, Runge – Kutta method, Discrete interval, Population equation, Non-linear population equation, Oscillation, Python.*

I. INTRODUCTION

In Numerical Methods, the Runge –Kutta methods are an important family of implicit and explicit iterative methods which are used in discretization for the approximation of solutions of ordinary differential equations [9]. These techniques were developed around 1900 by the German mathematicians Carle Runge and Martin Kutta. Runge –Kutta methods are applied to Simulation and games, Fuzzy differential equations, linear differential equations, non-linear differential equations, stochastic differential equations, uncertain differential equations and Schrodinger equations.

Xiangfeng Yang and Shen investigated the effectiveness of the Runge –Kutta method when calculating uncertainity distribution, expected value, extreme value and time integral solution of uncertain differential equations [1]. Qinghe Ming et al. presented the numerical results in the integration of Schrodinger equation to show the high efficiency of the new method [2]. J.T. Day developed a Runge –Kutta method for the numerical solution of hyperbolic partial differential equations [3]. G.U. Agebobah et al. developed the method of solving intial value problems in ordinary differential equations. Their results of Runge –Kutta formula generated through a FORTRAN program [4]. Faranak Rabiei et al. proposed the Runge –Kutta method for solving Volterra Integro differential equation [5].

Zhao Wenbo et al. developed to solve transient neutron diffusion equation using Euler's method and Runge –Kutta method. Their numerical evaluation shows that Runge –Kutta method is more accurate and efficient than Euler's method [6]. Andreas Robler introduced second order Runge –Kutta method for the weak approximation of the solution of Ito Stochastic differential equation systems with a multi-dimentional Wienner process [7]. This paper proposes the application of second order Runge –Kutta method for three different population initial value problems. The paper is organized as follows: Section II presents Second Order Runge – Kutta Method, Section III discusses the Population Equation, Section IV focuses on Implementation and Results and finally the Conclusion is presented in Section V.

II. SECOND ORDER RUNGE – KUTTA METHOD

The Runge –Kutta methods are designed to give greater accuracy and they possess the advantage of requiring only the function values at some selected points on the subinterval [10]. The general second order Runge – Kutta method for the first order differential equation

$y' = f(t,y)$ with the initial condition $y(t_0) = y_0 \dots\dots\dots\dots\dots..(1)$

is given by the formula

$$y_{i+1} = y_i + \frac{h}{2}[k_1 + k_2] \; for\; i = 0,1,2,\dots\dots\dots, N-1 \dots\dots\dots\dots\dots\dots (2)$$

where $k_1 = f(t_i\,, y_i)$

$k_2 = f(t_i + h\,, y_i + hk_1)$ and $h = interval\; length\; or\; step\; size$

III. POPULATION EQUATION

The general form of population growth differential equation is

$y' = ky \ldots\ldots\ldots\ldots..(3)$, where k is the growth rate.

The initial population at time a is $y(a) = A,\ a \le t \le b$

Integrating equation (3) gives the analytic solution $y = Ae^{kx}$. We will use this equation to illustrate the application of the Runge – Kutta method.

The general form of the non-linear sigmoidal population growth differential equation is

$$y' = \alpha y - \beta y^2 \ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

and the non-linear sigmoidal population growth differential equation with oscillation is

$$y' = \alpha y - \beta y^2 + \sin(2\pi t) \ldots\ldots\ldots\ldots\ldots\ldots.(5)$$

where α *is the growth rate and* β *is the death rate*. The initial population at time a is $y(a) = A,\ a \le t \le b$

1) *Specific Non-linear Population Equation:* Given the growth rate, $\alpha = 0.2$ *and death rate* $\beta = 0.01$, giving the specific non-linear population differential equation $y' = (0.2)y - (0.01)y^2$ and the specific non-linear population differential equation with oscillation $y' = (0.2)y - (0.01)y^2 + \sin(2\pi t)$. The initial population at time 2000 is $y(2000) = 6$, we are interested in the time period $2000 \le t \le 2020$.
2) *Initial Condition:* To obtain a specific solution to a first order initial value problem, the initial population is 6 billion people and therefore the initial condition is considered as $y(2000) = 6$. In the year 2000, the world population was 6.1143 billion.

Let us consider three initial value problems to apply the second order Runge – Kutta method.

a) *Linear Population Equation*

Consider the linear population differential equation

$$y' = (0.1)y, (2000 \le t \le 2020)$$

with the initial condition $y(2000) = 6$.

b) *Non-linear Population Equation*

Consider the non-linear population differential equation

$$y' = (0.2)y - (0.01)y^2, (2000 \le t \le 2020)$$

with the initial condition $y(2000) = 6$.

c) *Non-linear Population Equation with an Oscillation*

Consider the non-linear population differential equation with an oscillation

$$y' = (0.2)y - (0.01)y^2 + sin(2\pi t), (2000 \le t \le 2020)$$

with the initial condition $y(2000) = 6$.

3) *Discrete Interval:* The continuous time interval $a \le t \le b$ is discretized into N intervals separated by a constant step size $h = \frac{b-a}{N}$. Here the interval is $2000 \le t \le 2020$ with $N = 200$.

$$\therefore h = \frac{2020 - 2000}{200} = 0.1$$

This gives 201 discrete points with step size $h = 0.1$
$t_0 = 2000, t_1 = 2000.1, \ldots\ldots\ldots, t_{200} = 2020$.
This is generalized to $t_i = 2000 + (0.1)i,\ i = 0,1,2, \ldots\ldots\ldots\ldots..,200$.
The Figure 1 below shows the discrete time points for $h = 0.1$

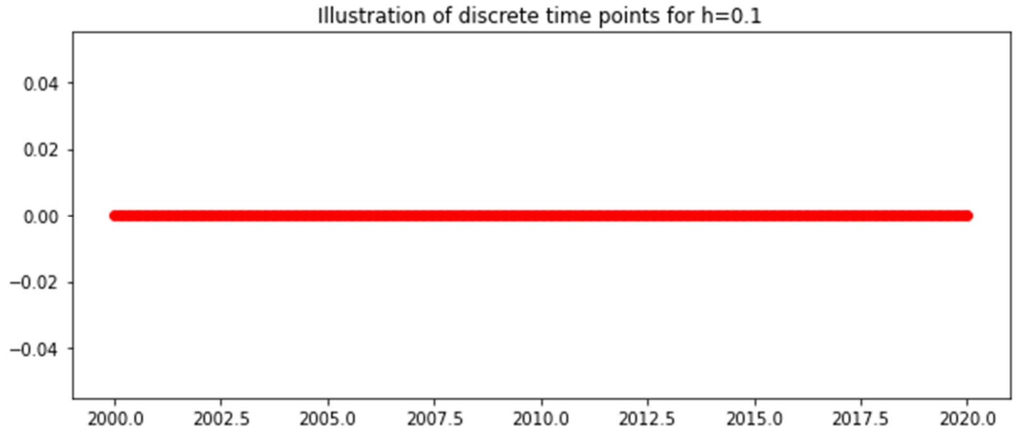


Fig. 1 Discrete time points for $h = 0.1$

IV. IMPLEMENTATION AND RESULTS

A. Runge – Kutta Method to Linear Population Equation

The linear population differential equation

$$y' = (0.1)y, (2000 \leq t \leq 2020) \ldots \ldots \ldots (6)$$

with the initial condition $y(2000) = 6$ has analytic solution $y = 6e^{(0.1)(t-2000)}$

To write the specific second order Runge – Kutta method for the linear population equation $f(t, y) = (0.1)y \ldots \ldots \ldots (7)$

which gives $k_1 = f(t_i, y_i) = (0.1)y_i$

$$k_2 = f(t_i + h, y_i + hk_1) = (0.1)(y_i + hk_1)$$

and the difference equation

$$y_{i+1} = y_i + \frac{h}{2}[k_1 + k_2] \; for \; i = 0,1,2, \ldots \ldots \ldots ,199 \ldots \ldots \ldots (8)$$

where y_i is the numerical approximation of y at time t_i with step size h and the initial condition $y_0 = 6$. The Figure 2 below shows the exact solution, y (squares) and the second order Runge – Kutta method numerical approximation y_i (circles) for the linear population equation.

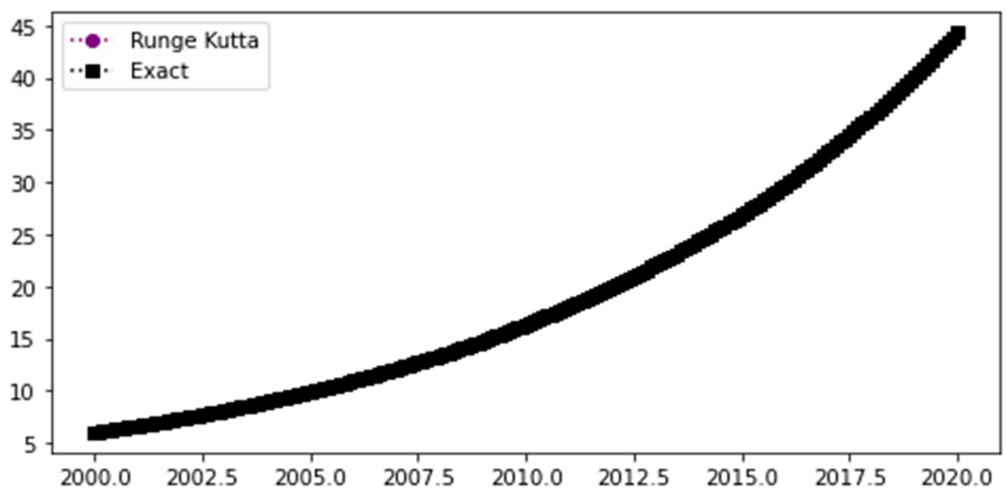


Fig. 2 Exact solution and Runge – Kutta approximation for Linear Population equation

Table 1 below shows the time, the second order Runge – Kutta numerical approximation y_i, the exact solution y and the exact error $|y(t_i) - y_i|$ for the linear population equation.

	time t_i	Runge Kutta	Exact (y)	Exact Error
0	2000.0	6.000000	6.000000	0.000000
1	2000.1	6.060300	6.060301	0.000001
2	2000.2	6.121206	6.121208	0.000002
3	2000.3	6.182724	6.182727	0.000003
4	2000.4	6.244861	6.244865	0.000004
5	2000.5	6.307621	6.307627	0.000005
6	2000.6	6.371013	6.371019	0.000006
7	2000.7	6.435042	6.435049	0.000007
8	2000.8	6.499714	6.499722	0.000009
9	2000.9	6.565036	6.565046	0.000010

Table 1.Runge – Kutta approximation to linear population equation

B. Runge – Kutta Method to Non-Linear Population Equation

Consider the non-linear population differential equation

$$y' = (0.2)y - (0.01)y^2, \qquad (2000 \le t \le 2020)$$

with the initial condition $y(2000) = 6$.

To write the specific second order Runge – Kutta method for the initial value problem we need

$$f(t, y) = (0.2)y - (0.01)y^2$$

This gives $k_1 = f(t_i, y_i) = (0.2)y_i - (0.01){y_i}^2$

$$k_2 = f(t_i + h, y_i + hk_1)$$

$$k_2 = (0.2)(y_i + hk_1) - 0.01(y_i + hk_1)^2$$

and the difference equation

$$y_{i+1} = y_i + \frac{h}{2}[k_1 + k_2] \, for \; i = 0,1,2, \dots \dots \dots ,199$$

where y_i is the numerical approximation of y at time t_i with step size h and the initial condition $y_0 = 6$. The Figure 3 below shows the second order Runge – Kutta method numerical approximation y_i (circles) for the non-linear population equation.

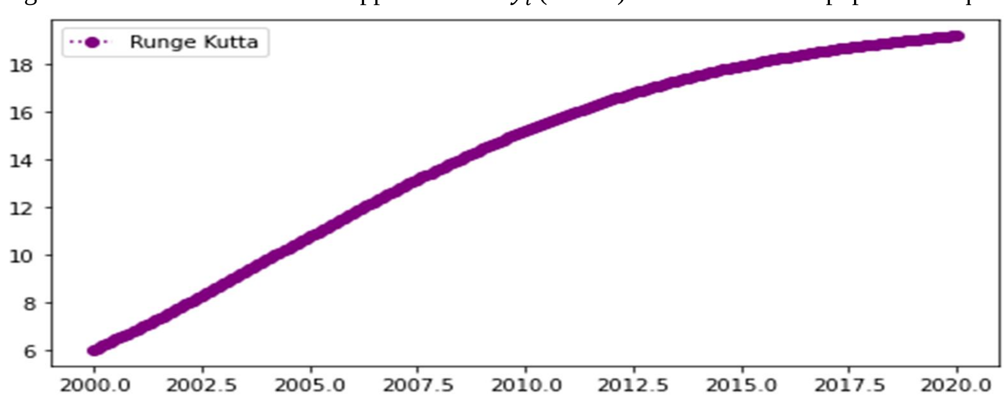


Fig. 3 Runge – Kutta approximation for non-linear population equation

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.538
Volume 10 Issue IV Apr 2022- Available at www.ijraset.com

The Table 2 below shows the time and the second order Runge – Kutta numerical approximation for the non-linear population equation.

	time t_i	Runge Kutta
0	2000.0	6.000000
1	2000.1	6.084332
2	2000.2	6.169328
3	2000.3	6.254977
4	2000.4	6.341270
5	2000.5	6.428197
6	2000.6	6.515747
7	2000.7	6.603909
8	2000.8	6.692672
9	2000.9	6.782025

Table 2 Runge – Kutta approximation to non-linear population equation

C. Runge – Kutta Method to Non-Linear Population Equation with an Oscillation

Consider the non-linear population differential equation with an oscillation

$$y' = (0.2)y - (0.01)y^2 + sin(2\pi t), \qquad (2000 \leq t \leq 2020)$$

with the initial condition $y(2000) = 6$.

To write the specific second order Runge – Kutta method for the initial value problem we need

$$f(t, y) = (0.2)y - (0.01)y^2 + \sin(2\pi t),$$

This gives $k_1 = f(t_i, y_i) = (0.2)y_i - (0.01){y_i}^2 + \sin(2\pi t_i)$

$$k_2 = f(t_i + h, y_i + hk_1)$$

$$k_2 = (0.2)(y_i + hk_1) - 0.01(y_i + hk_1)^2 + \sin(2\pi(t_i + h))$$

and the difference equation

$$y_{i+1} = y_i + \frac{h}{2}[k_1 + k_2]\ for\ i = 0,1,2, \dots \dots \dots, 199$$

where y_i is the numerical approximation of y at time t_i with step size h and the initial condition $y_0 = 6$. The Figure 4 below shows the second order Runge – Kutta method numerical approximation y_i (circles) for the non-linear population equation with an oscillation.

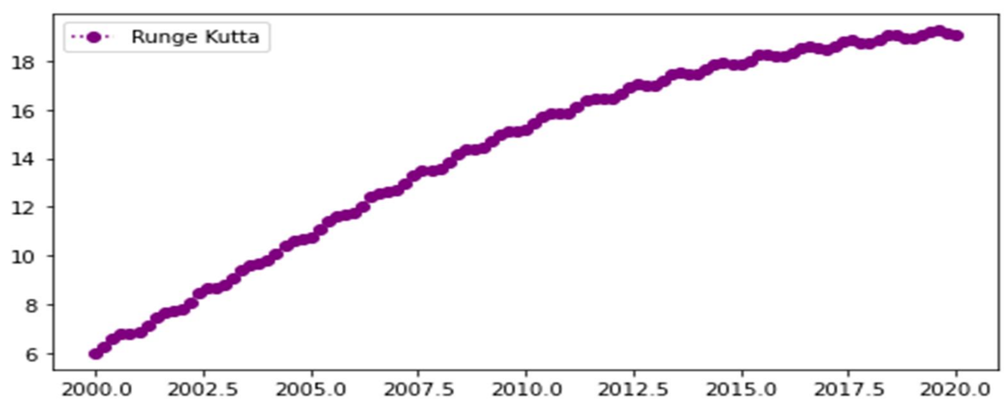


Fig. 4 Runge – Kutta approximation for non-linear population equation with oscillation

The Table 3 below shows the time and the second order Runge – Kutta numerical approximation for the non-linear population equation with oscillation.

	time t_i	Runge Kutta
0	2000.0	6.000000
1	2000.1	6.113722
2	2000.2	6.276109
3	2000.3	6.458005
4	2000.4	6.623032
5	2000.5	6.741504
6	2000.6	6.801784
7	2000.7	6.814712
8	2000.8	6.809444
9	2000.9	6.822305

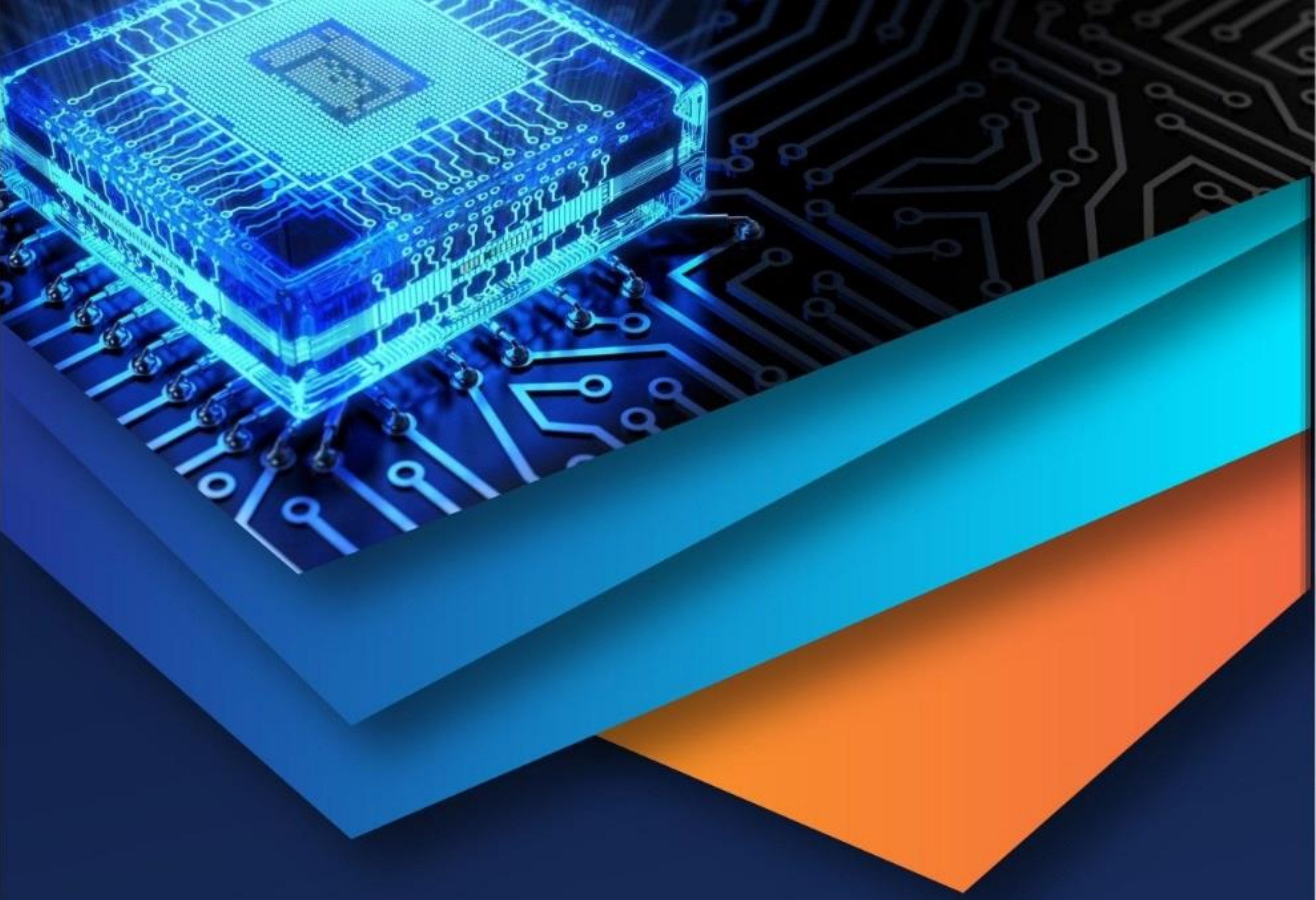
Table 3.Runge – Kutta approximation to non-linear population equation with an oscillation

V. CONCLUSION

We first introduced the second order Runge – Kutta method to the first order differential equation to obtain the numerical approximation of y at time t. We have proposed three different population initial value problems for linear population equation, non-linear population equation and non-linear population equation with an oscillation. To obtain the exact solution for the population equations, we have presented specific second order Runge – Kutta difference equation for the initial value problem. The time interval is discretized into N points by a constant step size. The solution is obtained by implementing Python programming for three initial value problems. The results are shown in Figure 2 for linear population equation and Table 1 shows the exact solution. Figure 3 and Table 2 shows the solution for non-linear population equation. Figure 4 and Table 3 shows the solution for non-linear population equation with an oscillation. In all three initial value problems, we observe that for Runge – Kutta approximation at time t shows the different population. The difference between population for linear population equation and non-linear population equation is 0.216989 billion approximately and that of linear population equation and non-linear population equation with an oscillation is 0.257269 billion approximately. The difference between population for non-linear population equation and non-linear population equation with an oscillation is 0.040280 billion approximately.

REFERENCES

[1] Xiangfeng Yang and Shen, “Runge – Kutta Method for Solving Uncertain Differential Equations”, Springer, Vol.3, Issue 17 (2015), pp. 01-12.
[2] Qinghe Ming et al., “An Optimized Runge – Kutta Method for the Numerical Solution of Schrodinger equation”, Mathematical Problems in Engineering, Hindawi Publishing Corporation, Volume 2012, pp 01-12.
[3] J.T.Day, “A Runge – Kutta Method for the Numerical Solution of the Goursat Problem in Hyperbolic Partial Differential Equations”, The Computer Journal, Vol. 9, Issue 1, May 1966, pp. 81-83.
[4] G.U. Agebeboh et al., “Implementation of New Fourth Order Runge – Kutta Formula for Solving Initial Value Problems”, International Journal of Physical Sciences, Vol. 2(4), April 2017, pp. 89-98.
[5] Faranak Rabiei et al., “Numerical Simulation of Fuzzy Volterra Integro – Differential Equation using Improved Runge – Kutta Method”, Journal of Applied and computational Mechanics, Vol. XX(X), 2021, pp. 01-11.
[6] Zhao Wenbo et al., “Application of Runge – Kutta Method to Solve Trancient Neutron Diffusion equation”, Atomic Energy Science and Technology,Vol. 47 (1), 2013, pp. 89-96.
[7] Andreas Robler, “Second Order Runge – Kutta Method for Ito Stochastic Differential Equation”, Society for Industrial and Applied Mathematics Journal on Numerical Analysis, Vol. 47, Issue 3, 2009, pp.1713-1738.
[8] S. S. Sastry, “Introductory Methods of Numerical Analysis” Third Edition, Prentice Hall of India, New Delhi.
[9] M. K. Jain et al., “Numerical Methods for Scientific and Engineering computation”, New Age International Publishers, Sixth Edition, 2014.
[10] Kendall E. Atkinson, “An Introduction to Numerical Analysis”, John Wiley & Sons.(1989).
[11] Stoer, J. and Bulirsch, R., “Introduction to Numerical Analysis”, Springer-Verlag (1980).



10.22214/IJRASET

45.98

IMPACT FACTOR: 7.129

IMPACT FACTOR: 7.429

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY