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Commutative Group with Respect to Neutrix Product

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Abstract: This research presents a comprehensive study of generalized functions and distribution theory, exploring their fundamental concepts, properties, and applications in mathematics and mathematical physics. The research aims to provide a thorough understanding of this powerful mathematical framework, shedding light on its versatile applications in diverse areas of study.
Keywords: Generalized functions, Commutative Group, Neutrix Product;

I. INTRODUCTION

Generalized functions and distributions, also known as generalized function theory or distribution theory, are mathematical tools used to extend the concept of functions beyond traditional functions that can be represented by ordinary mathematical expressions. They provide a framework for analysing and manipulating objects that are not necessarily well-defined functions in the classical sense.

Generalized functions and distribution theory emerged as a powerful mathematical framework in the mid-20th century to address the limitations of classical functions in describing certain phenomena and mathematical operations. Traditional functions often fail to capture singularities, impulses, and non-smooth phenomena encountered in various scientific and engineering disciplines. This necessitated the development of a more flexible and rigorous mathematical approach, leading to the formulation of generalized functions and distribution theory. In many scientific and engineering applications, traditional functions fail to capture certain phenomena or mathematical operations. For example, functions that describe point sources of energy or impulse-like events are difficult to represent using standard functions. Generalized functions offer a solution to this problem by providing a more flexible and powerful mathematical framework. The concept of generalized functions was first introduced by the French mathematician Laurent Schwartz in the 1940s. Schwartz developed the theory of distributions to provide a rigorous mathematical treatment of objects like the Dirac delta function, which represents an idealized point source. The theory of distributions was further developed and refined by many mathematicians, including Sergei Sobolev, Ivan Petrovsky, and Georges de Rham.

One key idea in distribution theory is that of a distribution as a linear functional acting on a space of test functions. Test functions are smooth and well-behaved functions with compact support, meaning they vanish outside a finite interval. Distributions can be thought of as generalized functionals that associate a value to each test function. This allows for the representation of objects like the Dirac delta function, which is not a conventional function but can be understood as a distribution.

Generalized functions and distributions find applications in various branches of mathematics, physics, and engineering. They are particularly useful in areas such as partial differential equations, Fourier analysis, signal processing, quantum mechanics, and general relativity. By employing the tools of distribution theory, researchers can handle singularities, impulses, and other non-smooth phenomena more rigorously and efficiently.

II. LITERATURE SURVEY

Ahuja and Prabha [1] tried to construct a commutative ring of generalized functions, but to construct examples, it is not so easy to define two binary operations simultaneously in a set of generalized functions to form a commutative ring.

J. G. Vander Corput [4] defined the neutrix N as a commutative additive group of functions $v(\xi)$ defined on a domain N’ with values in additive group N”, where further if for some v in N, $v(\xi) = \gamma$ for all ξ in N’, then $\gamma = 0$. the function in N are called negligible functions. Now let N’ be a set contained in a topological space with a limit point b which does not belong to N’. If $f(\xi)$ is a function defined on N’ with values in N” and it is possible to find a constant β such that $f(\xi) - \beta$ is negligible in N, then $\beta$is called the neutrix limit of N − limit of f as ξ tends to b and we write

$$N - \lim_{\xi \to b} (\xi) = \beta$$

where β must be unique, if it exists.
Fisher [2] has defined the neutrix product of two generalized functions as below:
Definition 1.1: Let f and g be arbitrary generalized functions as let

$$g_n = g * \delta_{n=} \int_{\frac{-1}{n}}^{\frac{1}{n}} g(x-t)\delta_n(t)dt$$

for n = 1, 2, 3, ... where δn = nρ (nx), ρ is an infinitely differentiable function satisfying the following properties:

$$(i)\ \rho(x) = 0\ for\ |x| \geq 1$$
$$(ii)\ \rho(x) \geq 0$$
$$(iii)\ \rho(x) = \rho(-x)$$
$$(iv)\ \int_{-1}^{1} p(x)dx = 1$$

We say that the matrix product f o g of f and g exists and equal to a generalized function h on (a, b) if

$$N - \lim_{n\to\infty}(fg_n, \emptyset) = N - \lim_{n\to\infty}(f, g_n\emptyset) = (h, \emptyset)$$

for all test functions ϕ with compact support contained in (a, b) where N is the neutrix having domain N' = (1, 2, ...n, ...) and range N" the real numbers with negligible functions linear sums of the functions

$$n^{\lambda} ln^{r-1} n, ln^{r} n$$

for λ > 0 and r = 1, 2, ... and all functions of n such that

$$\lim_{n\to\infty} f(n) = 0$$

The Neutrix product, also known as the Hadamard finite-part integral, is a mathematical operation that extends the concept of multiplication to distributions or generalized functions. It provides a way to multiply two distributions in a well-defined manner, even when their pointwise product is not well-defined due to singularities or other issues.
The Neutrix product is based on the notion of regularization and the concept of principal value. It involves subtracting the singular or divergent parts of the pointwise product of two distributions and taking the finite part of the result. This allows for a meaningful and consistent multiplication operation between distributions. Mathematically, if two distributions u(x) and v(x) are given, their Neutrix product, denoted as uжv, is defined as:

$$uжv = \lim_{\in\to 0}(u*(v*\rho\_\in) - u*(v*\rho^{-}\_\in))$$

where * represents the convolution operation, $\rho\_\in$ is a family of smooth approximation functions, and $\rho^{-}\_\in$ is their complement (often obtained by subtracting $\rho\_\in$ from a suitable constant). The limit is taken as ϵ approaches zero, capturing the finite part of the convolution result.
The convolution of two generalized functions is an operation that combines the effects of both functions to produce a new generalized function. It is defined as the integral of the product of one function with a translated and scaled version of the other function. Convolution is a fundamental operation in distribution theory and is used to define various operations and properties of generalized functions.
Convolution of distributions is an important operation in the theory of generalized functions. It allows us to combine the effects of two distributions to obtain a new distribution. The convolution operation is defined using the concept of the test function and the action of the distributions on these functions.

III.EXAMPLE

Let G = (x^{-r}: r is an integer), where the generalized function x^{-r} is defined in [3]. It can easily be seen the operation of neutrix product 'o' is defined in G. Now, we will show (G,o) is a commutative group.

1) *Closure Property Holds in G:* For x^{-p}, x^{-q} be arbitrary generalized functions in G

$$< x^{-p}\ o\ x^{-q}, \emptyset > = < x^{(-p)+(-q)}, \emptyset >$$

since (−p) + (−q) is again an integer we have

$$x^{(-p)+(-q)} \in G \forall x^{-p}, x^{-q} \in G$$

which shows that G is closed with respect to the neutrix product.

2) Associative Law Holds in G: We have

$$< x^{-p} \; o \; (x^{-q} \; o \; x^{-s}, \emptyset > = < x^{-p} \; o \; x^{(-q)+(-s)}, \emptyset >$$
$$= < x^{((-p)+(-q))\pm s}, \emptyset >$$
$$=< (x^{-p} \; o \; x^{-q}) \; o \; x^{(-s))}, \emptyset >$$
$$< x^{-p} \; o \; (x^{-q} \; o \; x^{-s}) > = (x^{(-p)} o x^{(-q)}) \; o \; x^{-s}$$

for all x^{-p}, x^{-q}, x^{-s} in G.

3) Identity Exists in G: There exists x0, known as identity in G such that
for all x^{-p} in G, we have

$$< x^{-p} \; o \; x^0, \emptyset > = < x^0 \; o \; x^{-p}, \emptyset > = < x^{(-p)}, \emptyset >$$
$$x^{-p} \; o \; x^0 = x^0 \; o \; x^{-p} = x^{-p}$$

4) Existence if Inverse in G: For each non-zero x^{-p} in G there exists a generalized function x^{-p} in G

$$< x^{-p} \; o \; x^p, \emptyset > = < x^p \; o \; x^{-p}, \emptyset > = < x^0, \emptyset >$$

and is known as the inverse of x^{-p} in G.

5) Commutative Law Holds in G: For all x^{-p}, x^{-q} in G we have

$$< x^{-p} \; o \; x^{-q}, \emptyset \geq < x^{(-p)+(-q)}, \emptyset >$$
$$< x^{(-q)+(-p)}, \emptyset > \;\; (since - p \; and - q \; are \; integers)$$
$$< x^{-q} \; o \; x^{-p}, \emptyset >$$
$$x^{-p} \; o \; x^{-q} = x^{-q} \; o \; x^{-p} \; \forall \; x^{-p} \; o \; x^{-q} \in G$$

Thus, (G, o) is a commutative group.

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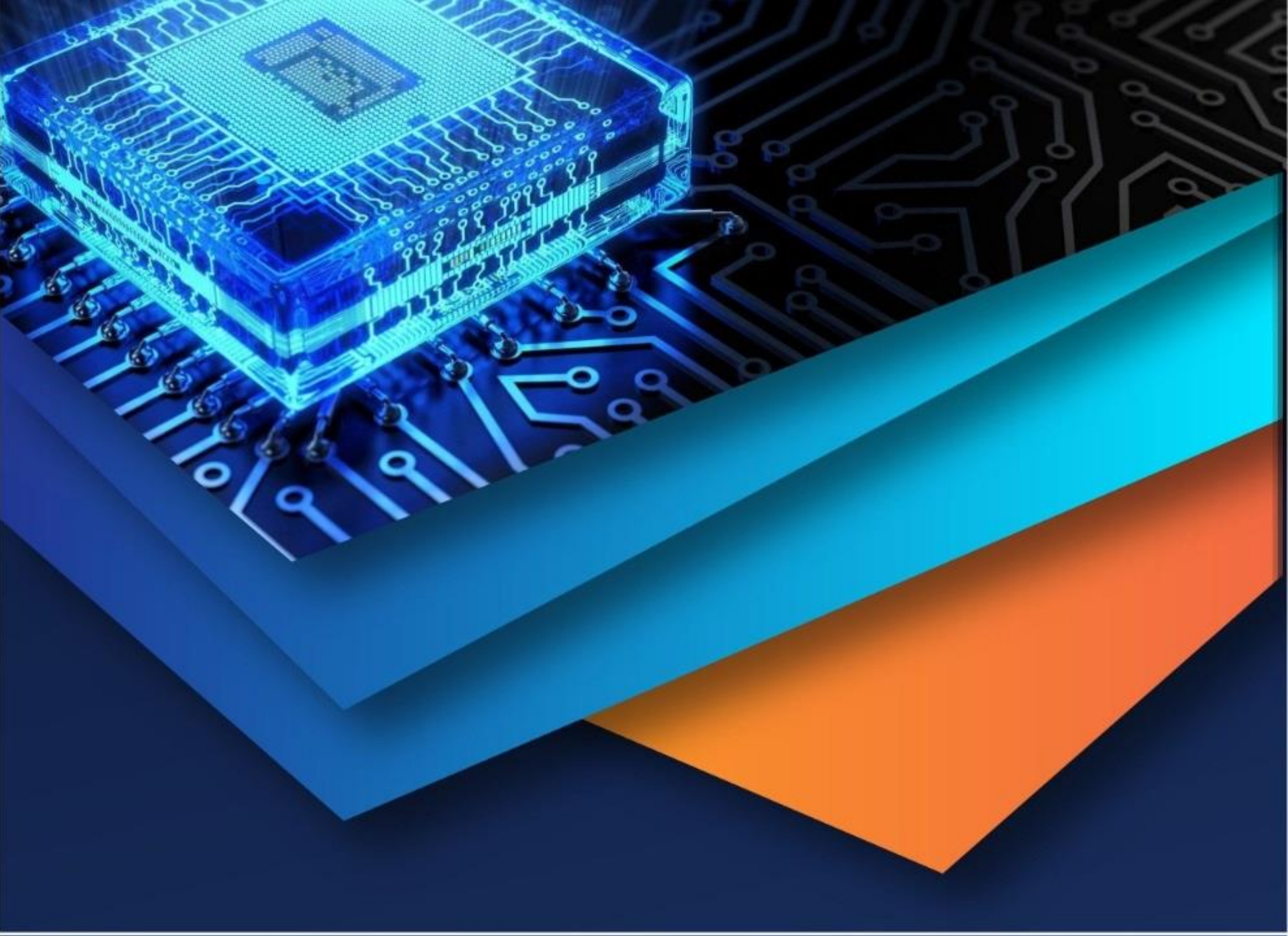

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