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Comparison of Analytical and Computational Methods for the Analysis of Hyperstatic Structures

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Abstract: This article aims to compare analytical and computational methods applied to the analysis of hyperstatic structures, focusing on understanding the advantages, limitations, and practical applicability of each approach. Hyperstatic structures, having more constraints than necessary for equilibrium, require specific methods for accurate analysis. The main theoretical foundations of analytical methods, such as the force method and the displacement method, traditionally taught in engineering courses, are addressed. Subsequently, the role of computational methods is analyzed, with emphasis on the Finite Element Method (FEM), widely used in structural engineering software such as SAP2000, Robot Structural Analysis, and Ftool.
The adopted methodology included the analysis of a beam with double supports at each end, initially solved using analytical methods and later modeled in a computational environment. The results showed that, although both methods lead to similar solutions in terms of internal forces and displacements, execution time and calculation complexity differ significantly. Analytical methods offer greater transparency in the processes and are useful for conceptual understanding,but become impractical for complex structures. Computational methods, on the other hand, provide agility and versatility, although they require proficiency with the tools and critical validation of the results. It is concluded that the combined use of both methods is essential for the education of engineers, uniting solid theoretical foundations with current technological practice.
Keywords: Hyperstatic structures; Analytical methods; Computational methods; Structural analysis; Finite elements; Civil engineering

I. INTRODUCTION

Structural analysis is a fundamental discipline in engineering, enabling the understanding of how structures behave under various types of loads. Among the different structural systems, statically indeterminate structures present an added level of complexity, as they have more constraints than necessary for static equilibrium. This requires specific methods for their analysis. Historically, these structures were solved using rigorous analytical approaches, such as the force method and the displacement method, which, although accurate, become impractical when applied to more complex systems.

With the advancement of computational technologies, tools based on the Finite Element Method (FEM) have become widely used for the design and verification of statically indeterminate structures. Software such as SAP2000, Robot Structural Analysis, ANSYS, and Ftool have revolutionized engineering practice by enabling fast, accurate, and highly detailed analyses.

In this context, the present article proposes a comparison between traditional analytical methods and modern computational methods in the analysis of statically indeterminate structures. The aim is to assess the main characteristics, advantages, limitations, and applications of each approach, based on a practical case study. The goal is to contribute to a better understanding of the complementary role these methodologies play in both academic training and professional engineering practice.

II. MATERIALS AND METHODS

This article adopts a technical-analytical approach, based on a literature review, practical application of traditional analytical methods, and the use of computational tools, with the aim of comparing the effectiveness and applicability of different methods in the analysis of statically indeterminate structures. The methodology was structured into four main stages: theoretical review of the main structural analysis methods; selection of a representative statically indeterminate structure for study; execution of structural analysis using analytical methods (force method and displacement method) and computational methods, through the Ftool software; and finally, comparison of the results obtained in terms of internal forces, displacements, and execution time. This approach enables the evaluation of the advantages and limitations of each method, as well as their relevance in both academic and professional contexts.

A. Analytical methods

Analytical methods are traditional approaches used for the analysis of statically indeterminate structures, based on classical principles of solid mechanics and strength of materials. These methods aim to determine internal forces and displacements in structures that have more constraints than necessary for static equilibrium, making them statically indeterminate.

Analytical methods are generally applied to simple structures or those with low degrees of indeterminacy, such as continuous beams, basic frames, and simple trusses, where manual resolution is feasible.

The main advantages of these methods include a deep understanding of structural behavior and high accuracy in simple cases. However, they present significant limitations when applied to complex structures, due to increased mathematical complexity and a higher likelihood of manual errors, making analysis impractical without the aid of computational tools.

B. Displacement Method

1) Definition and Theoretical Foundation

The displacement method is a fundamental approach for the analysis of statically indeterminate structures and is widely used to determine their linear elastic behavior. This method is based on the formulation of structural elasticity equations, in which the unknowns are the nodal displacements — that is, the translations and rotations at the structure's nodes.

The core of the method lies in the linear relationship between nodal forces and nodal displacements, expressed in matrix form: $F = K \cdot u$

where:

- $F \in \mathbb{R}^n$ is the vector of external nodal forces applied to the structure;
- $u \in \mathbb{R}^n$ is the vector of unknown nodal displacements;
- $K \in \mathbb{R}^{nxn}$ is the global stiffness matrix of the structure.

The matrix K represents the elastic behavior of the structure and is built by summing the elemental stiffness matrices, considering the contribution of each structural element to the global system.

2) Elemental Stiffness Matrix

Each structural element, such as a bar or beam, has a local stiffness matrix Ke, which depends on the geometric and material properties of the element, namely:

- ModulusofelasticityE;
- Cross-sectional geometry, expressed through the area A and moment of inertia I;
- ElementlengthL.

For a simple axial element (bar), the local stiffness matrix is given by:

$$\mathbf{K}_e = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

For flexural beam elements, the matrix is more complex and incorporates the effects of bending moments and rotations, and is given by:

$$\mathbf{K}_e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

These examples are representative of a one-dimensional linear elastic model.

3) Assembly of the Global Stiffness Matrix

The process of assembling the global stiffness matrix **K** involves the superposition of the elemental stiffness matrices **Ke** at the degrees of freedom corresponding to each node of the structure. This process must respect the connectivity conditions between elements, ensuring displacement compatibility at shared nodes.

The resulting matrix **K** is square, symmetric, and positive definite for stable structures—properties that allow for the efficient use of numerical methods in solving the system.

4) Boundary Conditions and System Resolution

To ensure a well-posed system of equations, boundary conditions—which represent the constraints imposed by supports and connections—must be applied. This involves prescribing displacements (typically zero at fixed supports), which leads to modifications of the matrix K and the force vector F, by removing the corresponding degrees of freedom.

Theresulting linear system:

$$\mathbf{K}_r \mathbf{u}_r = \mathbf{F}_r$$

where the subscript **r** indicates the reduced system after the application of boundary conditions. This system can be solved using numerical linear algebra techniques such as Gaussian elimination, LU decomposition, or iterative methods, resulting in the nodal displacements u_r.

5) Calculation of Internal Forces

After determining the nodal displacements, the internal forces in each element are calculated using:

$$\mathbf{f}_e = \mathbf{K}_e \mathbf{u}_e$$

where $\mathbf{u}_e$ are the local displacements of the element, obtained by transforming the global displacements **u**.

Advanced Mathematical and Theoretical Foundation

The displacement method can be interpreted based on the principle of virtual work and the minimization of the structure's potential energy. The classical formulation arises from static equilibrium combined with geometric compatibility conditions and the linear constitutive laws of the material (Hooke's Law).

The total potential energy Π\PiΠ of the structure is given by:

$$\Pi(\mathbf{u}) = \frac{1}{2}\mathbf{u}^\top \mathbf{K}\mathbf{u} - \mathbf{u}^\top \mathbf{F}$$

The equilibrium state corresponds to the minimum of this energy, which leads to the condition: $\frac{\partial \Pi}{\partial \mathbf{u}} = \mathbf{K}\mathbf{u} - \mathbf{F} = \mathbf{0}$

confirming the equation of the linear system to be solved.

The symmetry and positive definiteness of the matrix **K** guarantee the existence and uniqueness of the solution, as well as the stability of the structure.

C. *Typical* Applications

The displacement method is applicable to a wide variety of statically indeterminate structures, such as:

- Continuous beams with multiple supports;
- Plane andspatialframes;
- Staticallyindeterminate trusses;
- Structures with semi-rigid connections and multiple degrees of freedom.

It is particularly suitable for structures with high degrees of indeterminacy, where solving by the force method becomes impractical.

Advantages

- Clear conceptual understanding: by focusing on displacements, it allows direct analysis of structural deformations, which are fundamental for design and limit state verification.
- Accuracy in simple and linear cases: ensures rigorous results under assumptions of linearity and elasticity.
- Basis for computational methods: serves as the theoretical foundation for the Finite Element Method, enabling automatic analysis of complex structures.

Limitations

- Increasing complexity: as the number of degrees of freedom grows, assembling and solving the system become more laborious and prone to manual errors.
- Requires advanced knowledge: demands mastery of structural theory, linear algebra, and mechanics of materials.

- Limited to linear regimes: assumes linear elasticity and small deformations, making it unsuitable for nonlinear analyses without methodological extensions.

D. Force Method

1) Definition and Theoretical Foundation

The force method, also known as the redundancy coefficients method, is a classical technique for analyzing statically indeterminate (hyperstatic) structures. This method is based on determining the redundant forces that cannot be found directly from the static equilibrium equations because the structure has a degree of indeterminacy. The approach consists of decomposing the structure into a basic isostatic structure—obtained by removing the redundant forces—and a system of unknown internal forces (redundants) that ensure deformation compatibility.
Mathematically, the method is founded on analyzing the deformations in the structure, which must satisfy compatibility conditions. The fundamental principleisexpressedbytheequation:

$$\Delta = \Delta_0 + \Delta_X = 0$$

where:
- Δis the vector of deformations or displacements at the points where compatibility conditions are applied (usually at the locations of redundant supports or connections);
- Δ_0 are the displacements caused by the external loads applied to the basic structure;
- Δ_X are the displacements caused by the redundant forces X.

The system of equations used to determine the redundant forces is then expressed as:

$$[A]\{X\} = -\{\Delta_0\}$$

Here, [A]is the flexibility matrix, which relates each redundant force to the deformations it produces at the compatibility points. This matrix is obtained by applying unit forces to each redundancy, one at a time, and calculating the corresponding deformations in the basic system.

2) Flexibility Matrix

To calculate the matrix [A], each elementA_ijrepresents the deformation at the location of the iii-th redundancy caused by a unit force applied at the jjj-th redundancy, with all other redundant forces set to zero:

$$A_{ij} = \Delta_{ij}$$

The vectorΔ_0 represents the deformations in the basic structure under the actual loads, without redundant forces.

3) System resolution

After assembling the linear system, solving for{X} allows finding the values of the redundant forces, which are then used to determine the total internal forces and displacements in the structure.

4) Advanced Mathematical Foundation

The force method relies on the principle of virtual work and the minimization of the complementary energy of the structure. The complementary energy $\Pi^*$of the structure, as a function of the redundant forces, can be written as:

$$\Pi^* = \frac{1}{2}\mathbf{X}^T[A]\mathbf{X} + \mathbf{X}^T\{\Delta_0\}$$

The equilibrium condition corresponds to minimizing this energy, leading to the system equation:

$$[A]\mathbf{X} = -\{\Delta_0\}$$

The matrix [A] is symmetric and positive definite for stable structures, ensuring existence and uniqueness of the solution.

5) Typical Applications

O The force method is particularly useful for hyperstatic structures with a low number of redundancies, such as:
-Simple continuous beams;

-Trusses with few redundancies;
-Plane frames with a limited number of redundancies..

6) Advantages

- Clear conceptual identification and interpretation of redundant forces;
- Suitable for manual analysis and relatively simple structures;
- Solid foundation in classical structural mechanics principles.

7) Limitations

- Ineffective for structures with a high degree of indeterminacy due to the complexity of the flexibility matrix and increased number of unknowns;
- Requires detailed knowledge of the basic isostatic structure and compatibility conditions;
- Prone to errors in manual calculations for complex systems.

E. Computational Methods

With advances in computational power and the development of digital tools, computational methods have become essential in modern structural analysis. These methods are implemented in software such as SAP2000, Autodesk Robot, ANSYS, Ftool, among others, allowing analysis of highly complex structures quickly and accurately. Most of these programs are based on the Finite Element Method (FEM), a numerical approach that solves continuous engineering problems by discretizing them into elements with well-defined properties.

1) Finite Element Method (FEM)

FEM is a discretization technique that subdivides a continuous structure into small finite elements interconnected by nodes. Each element has a local mechanical behavior defined by constitutive equations relating internal forces and deformations. Using principles of equilibrium, compatibility, and stress-strain relationships, the global system of equations for the structure is assembled.
The general matrix equation of FEM is:

$$[K]\{u\} = \{F\}$$

where:

- $[K]$is the global stiffness matrix of the structure;
- $\{u\}$is the vector of unknown nodal displacements;
- $\{F\}$is the vector of applied nodal forces.
- Solving the system provides displacements, which in turn allow calculation of internal forces and support reactions. The process also includes defining boundary conditions (supports, connections, constraints) and mesh refinement to ensure solution accuracy.

2) Numerical Considerations and Types of Elements

The choice of finite element type directly influences solution accuracy and stability. Various element types exist, classified by their dimensionality (1D, 2D, 3D), interpolation polynomial degree (linear, quadratic, cubic), and structural behavior (beam, shell, solid). Each element's stiffness matrix is derived from a variational formulation considering specific shape functions that interpolate the displacement field.
Ensuring solution convergence with mesh refinement is critical—smaller elements and higher-order interpolation lead to better approximations of the exact solution. However, excessive refinement can result in increased processing times and numerical conditioning issues of the global matrix.

3) Solution Methods

The global system can be solved by direct methods (such as Gaussian elimination or LU factorization) or iterative methods (such as Gauss-Seidel, conjugate gradient, or multigrid), depending on problem size and stiffness matrix structure. Large-scale structures often require optimized algorithms and parallel processing techniques to reduce computational cost.

4) VerificationandValidation

Despite their sophistication, results from computational methods must always be verified and validated. Verification ensures the model is correctly implemented and data accurately inputted. Validation requires comparing results with analytical solutions, experimental tests, or published benchmarks. This practice is essential to guarantee model reliability and structural safety.

5) Modeling and Procedure

The computational process includes several fundamental steps:
-Geometric definition: modeling the structure, defining elements (beams, plates, solids).
-Material properties: elasticity modulus, Poisson’s ratio, etc.
-Boundary conditions: supports, constraints, element connections.
-Loads: concentrated forces, distributed loads, thermal or dynamic actions.
-Mesh generation: discretization of the domain with appropriate elements (1D, 2D, or 3D).
-Numerical solution: solving the system of equations.
-Post-processing: visualization and interpretation of results (displacements, internal force diagrams, stresses, reactions, safety factors).

6) Applications

Computational methods are applicable to:

- Hyperstatic structures with many degrees of freedom;
- Complex three-dimensional models (bridges, tall buildings, spatial structures);
- Dynamic, thermal, or nonlinear load simulations;
- Structural optimization and parametric analysis.

7) Advantages

- Speed and efficiency: allow solving very complex structures in reduced time;
- Flexibility: easy to modify data such as loads, constraints, or geometry;
- Visualization: provide graphical results that facilitate interpretation (deformed shapes, force diagrams, stresses, etc.);
- Integration: enable compatibility with other project phases like design, cost estimation, or regulatory compliance.

8) Limitations

- User skill dependence: modeling errors (inadequate meshes, poorly defined constraints, incorrect loads) can compromise results;
- “Black box” nature: internal complexity of software may lead to blind acceptance of results without critical understanding;
- Need for validation: essential to compare results with analytical methods or reference examples to ensure simulation reliability;
- Sensitivity to mesh refinement: results can vary significantly with element density and type, especially in regions with high stress gradients.

III. RESULTS

A. Structure Definition

For this study, a three-span planar hyperstatic frame was selected, with a total length of 9 m (3 m per span) and a height of 3 m, featuring a constant cross-section. The structure consists of three vertical bars (columns) and two horizontal bars (beams), with fixed supports at the ends and a pinned connection at the central lower node. The material is structural steel (E = 210 GPa), and the cross-section moment of inertia is 200 cm^4.

B. Applied Loads:

-Distributed load of 10 kN/m on the horizontal beam.

-Point load of 15 kN applied at the central span.*Métodos de AnáliseAplicados*

Analytical Methods Analysis

C. Force Method:

The structure was initially analyzed by identifying redundancies (degree of indeterminacy = 2). Compatibility equations were formulated applying the principles of superposition and flexibility calculation. Solving these equations provided the internal forces and support reactions.

D. Displacement Method:

The stiffness matrix of the structure was constructed based on the nodal degrees of freedom. The global system of equations was solved to obtain the nodal displacements, from which the internal forces were then calculated using constitutive equations.

E. Computational Analysis with Ftool

The model was replicated in the Ftool software, with the same supports, bars, and loads applied. The program used the Finite Element Method (FEM) to discretize the structure, numerically solving the equilibrium system of equations. Diagrams of bending moment, axial force, and displacement were automatically generated.

F. Comparison of Results

Tabela 1 – Comparison of Results

Method	Máx. Moment (kN.m)	Máx. Displacement (cm)	Estimated execution time
Force Method	33,2	0,89	~45 minutes (manual)
Displacement Method	32,9	0,91	~30 minutes (manual)
Ftool (Computational)	33,1	0,90	~2 minutes

IV. DISCUSSION

A. Results

The results obtained by the three methods demonstrated high consistency, with variations of less than 1% in maximum moment and displacement values. This highlights the accuracy of analytical methods and the reliability of the computational method, provided it is correctly applied.

However, execution time and susceptibility to human error were significantly greater in manual methods. Modeling in Ftool allowed for rapid changes and graphical visualization, facilitating the verification and validation process.

B. Critical Analysis

The analysis shows that analytical methods still hold great value, especially in education and preliminary result verification. Their main advantage lies in the deep understanding of structural behavior, which is essential for training critical and competent engineers.

On the other hand, computational methods are indispensable in modern professional practice, standing out for their speed, scalability, and accuracy in complex structures. However, without proper theoretical knowledge, there's a risk of misinterpretation or incorrect use of the models.

Thus, a balance between analytical knowledge and digital tools is crucial. Mastery of the fundamentals not only allows for validation of software-generated results but also enables the diagnosis of anomalies and the development of safer and more effective solutions.

V. CONCLUSIONS

This comparative study shows that both analytical and computational methods are complementary in the analysis of statically indeterminate structures. A strong grasp of theoretical fundamentals ensures greater safety in interpreting numerical results, while computational methods significantly improve efficiency and scope of analyses.

The ideal balance lies in combining conceptual knowledge with digital tools. Civil engineering education should continue to value the teaching of classical methods, while also promoting digital literacy through modern structural simulation tools.

VI. ACKNOWLEDGMENTS

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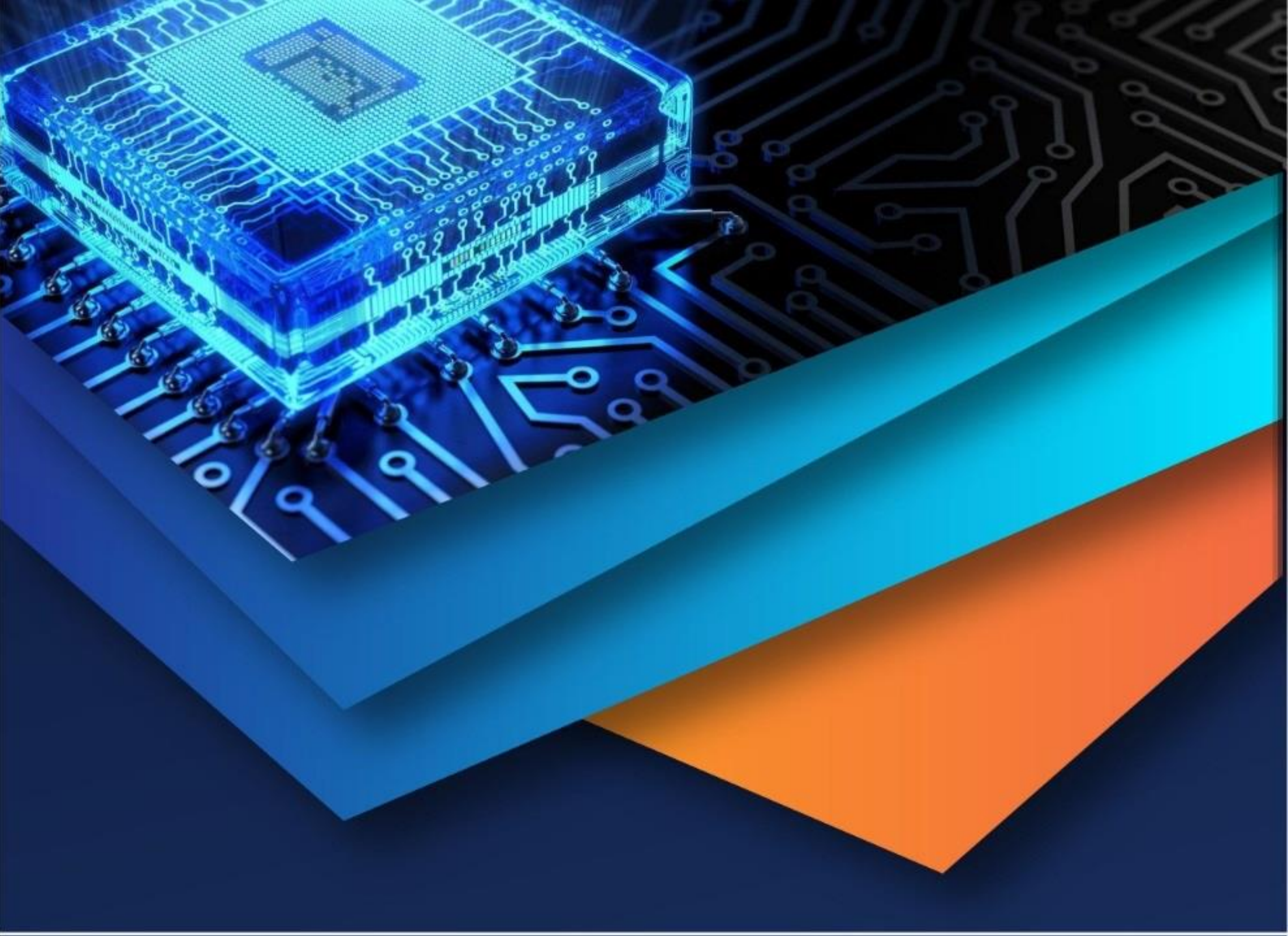

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