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Continuation Method for Multiparameter Eigenvalue Problem

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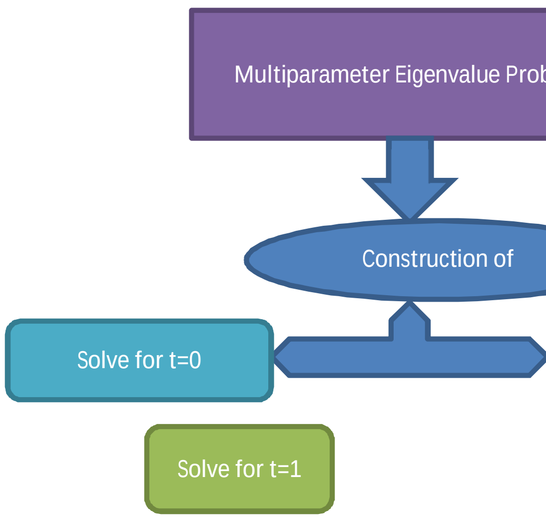
Abstract: Differential equations often arise in the mathematical formulation of problems in science and engineering. When solving certain boundary value problems in partial differential equations using the separation of variables method, one often encounters multiparameter eigenvalue problems. There has been limited numerical research on multiparameter eigenvalue problems, making further study an urgent necessity. This paper focuses on obtaining numerical solutions to multiparameter eigenvalue problems through the continuation method. Finally, numerical results are presented to illustrate the effectiveness of the proposed method.
Keywords: Multiparameter, Eigenvalue, Eigenvector,Homotropy, Continuation, Convergence.

Graphical Abstract:



I. INTRODUCTION

The multiparameter eigenvalue problem generalizes the one-parameter eigenvalue problem. In many areas of mathematical physics multiparameter eigenvalue problems are arised. Some examples of multiparameter eigenvalue problems are a dynamical problem of a homogeneous beam loaded by a vertical load Collatz [2], a vibrating membrane problem Roach [4], the divergent flow problem between non-parallel planes in an electrically conducting fluid under a transverse magnetic field Srivastava and Hazarika [5] etc. Such problems occur when the method of separation of variables is employed to solve boundary value problems derived from partial differential equations Volkmer [6].

A multiparameter eigenvalue problem in matrix form is as follows

$$M_1 u_1 = (A_{10} - \lambda_1 A_{11} - \lambda_2 A_{12} - \cdots - \lambda_n A_{1n}) u_1 = 0$$
$$M_2 u_2 = (A_{20} - \lambda_1 A_{21} - \lambda_2 A_{22} - \cdots - \lambda_n A_{2n}) u_2 = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1.1.1)$$

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$$M_n u_n = (A_{n0} - \lambda_1 A_{n1} - \lambda_2 A_{n2} - \cdots - \lambda_n A_{nn}) u_n = 0$$

where $\lambda_i \in C, i = 1,2,3,\ldots,n$

$u_i \in C^n\backslash\{0\}, i = 1,2,\ldots,n;\ A_{ij} \in C^{n\times n}, i = 1,2,\ldots,n \ and\ j = 0,1,2,\ldots,n$. Here we consider A_{ij} are symmetric $n_i \times n_i$ matrices over R, $i = 1,2,3,\ldots,n$ and $j = 0,1,2,3,\ldots,n$.

For all vectors $\|u_1\| = \|u_2\| = \cdots = \|u_n\| = 1$, we consider

$$D(u_1, u_2, \ldots, u_n) = \begin{vmatrix} {u_1}^T A_{11} u_1 & \ldots & {u_n}^T A_{1n} u_n \\ \vdots & \ldots. & \vdots \\ {u_n}^T A_{n1} u_n & \cdots & {u_n}^T A_{nn} u_n \end{vmatrix} \geq \delta > 0 \qquad (1.1.2)$$

The problem (1.1.1) is called right definite Atkinson [4] if the condition (1.1.2) holds. The problem (1.1.1) is called as diagonal right definite if the matrices $A_{11}, A_{22}, \ldots, A_{nn}$ are positive definite.

The equation (1.1.1) can be reduced to a system of n one-parameter eigenvalue problems.

$$\begin{aligned} \Delta_1 u &= \lambda_1 \Delta_0 u \\ \Delta_2 u &= \lambda_2 \Delta_0 u \\ &\vdots \\ \Delta_n u &= \lambda_n \Delta_0 u \end{aligned} \qquad (1.1.3)$$

where $u = u_1 \otimes u_2 \otimes \ldots \ldots \otimes u_n$

$$\text{and } \Delta_0 = \begin{vmatrix} \tilde{A}_{11} & \ldots.. & \tilde{A}_{12} & \tilde{A}_{1r} & \tilde{A}_{1\,r+1} & \ldots.. & \tilde{A}_{1n} \\ \tilde{A}_{21} & \ldots.. & \tilde{A}_{22} & \tilde{A}_{2r} & \tilde{A}_{2\,r+1} & \ldots.. & \tilde{A}_{2n} \\ \ldots.. & \ldots.. & \ldots.. & \ldots.. & \ldots.. & \ldots.. & \ldots.. \\ \tilde{A}_{n1} & \ldots.. & \tilde{A}_{n2} & \tilde{A}_{nr} & \tilde{A}_{n\,r+1} & \ldots.. & \tilde{A}_{nn} \end{vmatrix} \qquad (1.1.4)$$

$$\Delta_r = \begin{vmatrix} \tilde{A}_{11} & \ldots.. & \tilde{A}_{1\,r-1} & \tilde{A}_{10} & \tilde{A}_{1\,r+1} & \ldots.. & \tilde{A}_{1n} \\ \tilde{A}_{21} & \ldots.. & \tilde{A}_{2\,r-1} & \tilde{A}_{20} & \tilde{A}_{2\,r+1} & \ldots.. & \tilde{A}_{2n} \\ \ldots.. & \ldots.. & \ldots.. & \ldots.. & \ldots.. & \ldots.. & \ldots.. \\ \tilde{A}_{n1} & \ldots.. & \tilde{A}_{n\,r-1} & \tilde{A}_{n0} & \tilde{A}_{n\,r+1} & \ldots.. & \tilde{A}_{nn} \end{vmatrix} \qquad (1.1.5)$$

where $\tilde{A}_{1j}, \tilde{A}_{2j}, \ldots. \tilde{A}_{nj}$ are operators in H induced by operators $A_{1j}, A_{2j}, \ldots., A_{nj}, j = 0,1,2,\ldots,n$.

or we can write

$$\begin{aligned} \tilde{A}_{10} &= A_{10} \otimes I_2 \otimes \ldots\ldots.. \otimes I_n, & \tilde{A}_{1r} &= A_{1r} \otimes I_2 \otimes \ldots\ldots.. \otimes I_n \\ \tilde{A}_{20} &= I_1 \otimes A_{20} \otimes \ldots\ldots.. \otimes I_n, & \tilde{A}_{2r} &= I_1 \otimes A_{2r} \otimes \ldots\ldots.. \otimes I_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ \tilde{A}_{n0} &= I_1 \otimes I_2 \otimes \ldots\ldots. \otimes A_{n0}, & \tilde{A}_{nr} &= I_1 \otimes I_2 \otimes \ldots\ldots.. \otimes A_{nr} \end{aligned}$$

where I_r is the identity operator in H_r, $r = 1,2,\ldots,n$.

For decomposable tensors we may refer Atkinson [1].

A. *Objective of the Study*

Main objective of this paper is to find the numerical solution of multiparameter eigenvalue problem.

II. HOMOTOPY CONTINUATION METHOD

Homotopy continuation methods are powerful techniques that make significant contributions to the numerical solution of nonlinear systems of equations." Authors like Allgower and Georg [2, 3], Chu [7], Chu, Li and Sauer [8], Hlavacek and Seydel [9], Li and Rhee [10], Lindfield and Simpson [11], Miersemann and Mittelmann [12], Morgan [13] have dealt with Homotopy method.

In homotopy methods, a simpler parameter-dependent function $H(x,t)$, simpler than the original one is constructed.. The fundamental idea of the continuation method is to determine the solution at at $t_0 + \delta t$ for a small change δt, given that the solution if the solution of $(x_0, t_0)$is known.

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In order to solve a system of N non-linear equations in N-variables, denoted as $F(x) = 0$, using Homotopy Continuation method, one first constructs a parameter depending function $H(x,t)$, where $t \in [0,1]$ such that $H(x,0) = 0$ represents a problem with a known solution, and $H(x,1) = 0$ corresponds to the original problem $F(x) = 0$.

In the continuation method, the parameter t is discretizeed as $0 = t_0 < t_1 < \cdots < t_n = 1$ where $t(0) = t_0, t_1 = t_0 + h, t_2 = t_1 + h$ ....and the solution of $H(x,t) = 0$ for $t = 0$, serves as the starting point for computing the solution at $t = t_1$. Repeating this process step by step yields the solution at $t = 1$, which solves the original system $F(x) = 0$.

The continuation method has been successfully applied to both standard $AX = \lambda X$ and generalized $AX = \lambda BX$ eigenvalue problems. Chu [7] investigated the homotopy approach for symmetric matrices, and Li and Rhee [10] demonstrated its practical use. Simasaki[15] and Plestenjak[14] further extended the continuation method to two-parameter eigenvalue problems, including right-definite and general right-definite cases. In this study, the continuation method is applied to right-definite multi-parameter eigenvalue problems.

III. HOMOTOPY CONTINUATION METHOD FOR MULTI-PARAMETER EIGENVALUE PROBLEMS

In this section we consider the matrices $A_{11}, A_{22}, \dots, A_{nn}$ in (1.1.1) are positive definite and following Plestenjak [14] we construct the following Homotopy

$H: R^{n_1} \times R^{n_2} \times \dots \times R^{n_n} \times R \times R \times R \dots \times R \times [0,1] \to R^{n_1} \times R^{n_2} \times \dots \times R^{n_n} \times R \times R \times \dots \times R$ as

$$H(u_1, u_2, .. u_n, \lambda_1, \lambda_2, .., \lambda_n, t) = \begin{bmatrix} (1-t)W_1u_1 + tA_{10}u_1 - \lambda_1 A_{11}u_1 - \lambda_2 tA_{12}u_1 - \cdots - \lambda_n tA_{1n}u_1 \\ \vdots \\ (1-t)W_n u_n + tA_{n0}u_n - \lambda_1 tA_{n1}u_n - \lambda_2 tA_{n2}u_n - \cdots - \lambda_n A_{nn}u_n \\ \frac{1}{2}(1 - {u_1}^T u_1) \\ \vdots \\ \frac{1}{2}(1 - {u_n}^T u_n) \end{bmatrix} \qquad (1.3.1)$$

where W_i are symmetric $n_i \times n_i$ $(i = 1,2,3, \dots, n)$ matrices, such that the eigenvalue problem

$$W_1 u_1 = \lambda_1 A_{11} u_1$$
$$\cdots\cdots\cdots \qquad (1.3.2)$$
$$W_n u_n = \lambda_n A_{nn} u_n$$

have $n_1, n_2, \dots, n_n$ distinct eigenvalues respectively.

A solution of $H(u_1, u_2, .. u_n, \lambda_1, \lambda_2, .., \lambda_n, t) = 0$ is a solution of the multi-parameter eigenvalue problem

$$(1-t)W_1u_1 + tA_{10}u_1 = \lambda_1 A_{11}u_1 + \lambda_2 tA_{12}u_1 + \cdots + \lambda_n tA_{1n}u_1$$
$$\vdots \qquad (1.3.3)$$
$$(1-t)W_n u_n + tA_{n0}u_n = \lambda_1 tA_{n1}u_n + \lambda_2 tA_{n2}u_n + \cdots + \lambda_n A_{nn}u_n$$

For $t = 1$, (1.3.3) is equal to (1.1.1) and for $t = 0$, (1.3.3) is equal to (1.3.2).

Consider the operator determinant for the problem (1.3.3) as

$$\Delta_0(t) = \begin{vmatrix} A_{11}^\dagger & tA_{12}^\dagger \dots\dots\dots & tA_{1n}^\dagger \\ \vdots & \vdots \cdots \dots\dots.. & \vdots \\ tA_{n1}^\dagger & tA_{n2}^\dagger \dots\dots\dots.. & A_{nn}^\dagger \end{vmatrix}$$

$$\Delta_1(t) = \begin{vmatrix} (1-t)W_1^\dagger + tA_{10}^\dagger & tA_{12}^\dagger \dots. & tA_{1n}^\dagger \\ \vdots & \dots\dots. & \vdots \\ (1-t)W_n^\dagger + tA_{n0}^\dagger & tA_{n2}^\dagger \dots.. & A_{nn}^\dagger \end{vmatrix}$$

$$\Delta_2(t) = \begin{vmatrix} A_{11}^\dagger & (1-t)W_1^\dagger + tA_{10}^\dagger \dots & tA_{1n}^\dagger \\ \vdots & \dots\dots\dots\dots\dots & \vdots \\ tA_{n1}^\dagger & (1-t)W_n^\dagger + tA_{n0}^\dagger \dots. & A_{nn}^\dagger \end{vmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta_n(t) = \begin{vmatrix} A_{11}^\dagger & tA_{12}^\dagger \dots & (1-t)W_1^\dagger + tA_{10}^\dagger \\ \vdots & \dots\dots & \vdots \\ tA_{n1}^\dagger & tA_{n2}^\dagger \dots & (1-t)W_n^\dagger + tA_{n0}^\dagger \end{vmatrix}$$

where † denotes the induced linear transformation.

Consider $D(u_1, u_2, \dots, u_n, t) = \begin{vmatrix} {u_1}^T A_{11} u_1 & \dots & t{u_n}^T A_{1n} u_n \\ \vdots & \dots . & \vdots \\ t{u_n}^T A_{n1} u_n & \cdots & {u_n}^T A_{nn} u_n \end{vmatrix}$

and for arbitrary vectors $u_1, u_2, \dots, u_n$ and $t \in [0,1]$, we have

$$\min\{D(u_1, u_2, \dots, u_n, 0),\ D(u_1, u_2, \dots, u_n, 1)\} \le D(u_1, u_2, \dots, u_n, t)$$

The problem (1.3.3) is right definite for all $t \in [0,1]$ due to Plestenjak [14].

Here we discretize t as $0 = t_0 < t_1 < \cdots < t_n = 1$ where $t(0) = t_0$, $t_1 = t_0 + h$, $t_2 = t_1 + h$ .... and so on. First one has to solve (1.3.1) for $t = 0$ i.e the problem (1.3.2) to obtain the initial eigenvalue $\lambda_1(0)$, $\lambda_2(0)$, $, \dots, \lambda_n(0)$ and initial eigenvector $u_1(0)$, $u_2(0)$, $u_n(0)$, then taking these as the initial approximation for $t = t_1$. Using Newton's method solving the problem (1.3.1) at $t = t_1$. Taking the solution at $t = t_1$ as an initial approximation, we solve the problem for $t = t_2$ using Newton's method and so on and finally we will obtain a solution for $t = 1$ and it will give us the solution of (1.1.1).

IV. NUMERICAL EXAMPLE

A numerical example for Homotopy Continuation method is discussed in this section.

Consider the following three-parameter eigenvalue problem in matrix form.

$$\begin{aligned}
&\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}x-\lambda_1\begin{bmatrix}2&1&1\\1&3&-.5\\1&-.5&2\end{bmatrix}x-\lambda_2\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}x-\lambda_3\begin{bmatrix}4&1&2\\1&5&0\\2&0&4\end{bmatrix}x=0\\
&\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}y-\lambda_1\begin{bmatrix}1&1&1\\1&0&1\\1&1&1\end{bmatrix}y-\lambda_2\begin{bmatrix}3&1&3\\1&3&1\\3&1&4\end{bmatrix}y-\lambda_3\begin{bmatrix}6&1&-.5\\1&6&1\\-.5&1&7\end{bmatrix}y=0 \qquad (1.4.1)\\
&\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}z-\lambda_1\begin{bmatrix}2&-.5&0\\-.5&3&1\\0&1&2\end{bmatrix}z-\lambda_2\begin{bmatrix}1&1&1\\1&1&1\\1&1&3\end{bmatrix}z-\lambda_3\begin{bmatrix}1&1&0\\1&2&1\\0&1&2\end{bmatrix}z=0
\end{aligned}$$

Let us consider the Homotopy

$$\begin{aligned}
&(1-t)W_1x + tA_{10}x = \lambda_1 A_{11}x + t\lambda_2 A_{12}x + t\lambda_3 A_{13}x\\
&(1-t)W_2y + tA_{20}y = \lambda_1 tA_{21}y + \lambda_2 A_{22}y + t\lambda_3 A_{23}y \qquad (1.4.2)\\
&(1-t)W_3z + tA_{30}z = \lambda_1 tA_{31}z + t\lambda_2 A_{32}z + \lambda_3 A_{33}z
\end{aligned}$$

Let $W_1 = \begin{bmatrix}0&1&1\\1&1&1\\1&1&1\end{bmatrix}$, $W_2 = \begin{bmatrix}1&1&1\\1&0&1\\1&1&1\end{bmatrix}$, $W_3 = \begin{bmatrix}1&1&1\\1&1&1\\1&1&0\end{bmatrix}$

For $t = 0$ equation (5.4.2) becomes

$$\begin{aligned}
W_1x &= \lambda_1 A_{11}x\\
W_2y &= \lambda_2 A_{22}y \qquad (1.4.3)\\
W_3z &= \lambda_3 A_{33}z
\end{aligned}$$

For $t = 1$, the equation (1.4.2) is equivalent to the equation (1.4.1). Here we consider $h = 0.2$ and discretize t as $t_0 = 0$, $t_1 = t_0 + h = 0.2$, $t_2 = t_1 + h = 0.4$, $t_3 = t_2 + h = 0.6$, $t_4 = t_3 + h = 0.8$, $t_5 = t_4 + h = 1$.

Taking the solutions of (1.4.3) as initial approximation we will solve the given problems using Newton's method for $t = t_1$. Again taking the solution of (1.4.2) at $t = t_1$ as initial approximation for the problem (1.4.2) at $t = t_2$ and so on and finally the problem (1.4.2) at $t = t_5 = 1$ will give us the eigenvalues for the problem (1.4.1).

The solutions of (1.4.3) are

$\lambda_1 = (-1.0445, 1.0445, 0)$ and the corresponding eigenvectors are $(0.8067, -0.3435, -0.4808)$, $(0.0297, -0.5810, -0.8134)$, $(0, -0.7071, 0.7071)$

$\lambda_2 = (0.42155, -0.2965, 0)$ and the corresponding eigenvectors are $(0.9094, 0.4160, 0)$, $(-0.5658, 0.8246, 0)$, $(-0.7071, 0, 0.7071)$

$\lambda_3 = (1.6180, -0.6180, 0)$ and the eigenvectors are $(0.8798, -0.3361, 0.3361)$, $(0.2607, -0.6826, 0.6826)$, $(-0.7071, 0.7071, 0)$

Taking $\lambda_1(0) = 1.0445$, $\lambda_2(0) = -0.2965$, $\lambda_3(0) = 0$

and $x(0) = (0.0297, -0.5810, -0.8134)$, $y(0) = (-0.5658, 0.8246, 0)$,

$z(0) = (-0.7071, 0.7071, 0)$ as initial approximation we have the following eigenvalue curve

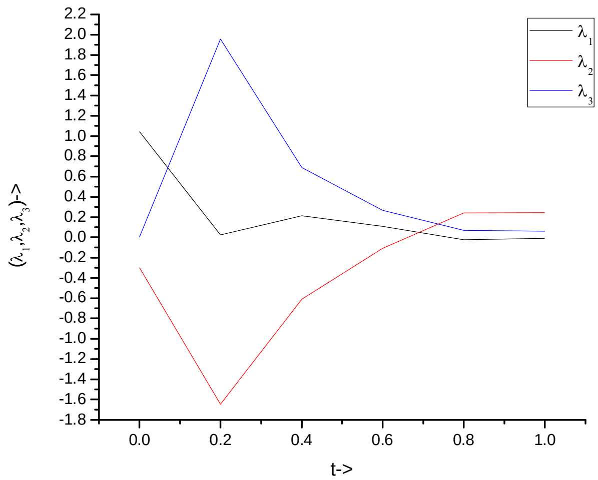

Fig. 1.4.1: Eigenvalue curve of the problem (1.4.1)

Above figure gives the graph of $\lambda_1(t), \lambda_2(t), \lambda_3(t)$. Here stepsize is taken $h = 0.2$. For better result we may decrease the stepsize. So in this way taking different starting eigenvalues we can trace the eigencurves and study the behaviour of the curves. The following table gives the different eigenvalues obtained by taking all the possible initial eigenvalues.

Table 1.4.1: Estimate of eigenvalues for different initial eigenvalues

$Sl.no$	t	$(\lambda_1, \lambda_2, \lambda_3)$
1	$t = 0$	$(1.0445, 0.4215, 0)$
	$t = 0.2$	$(-0.1463, -1.3101, 2.1436)$
	$t = 0.4$	$(0.0788, 0.0392, -0.1197)$
	$t = 0.6$	$(-0.0029, 0.0318, 0.0102)$
	$t = 0.8$	$(-0.0025, 0.0436, 0.0117)$
	$t = 1$	$(2.7867e-007, -6.8887e-006, -1.7106e-006)$
2	$t = 0$	$(1.0445, -0.2965, 0)$
	$t = 0.2$	$(0.0251, -1.6458, 1.9573)$
	$t = 0.4$	$(0.2127, -0.6091, 0.6891)$
	$t = 0.6$	$(0.1097, -0.1077, 0.2676)$
	$t = 0.8$	$(-0.0232, 0.2415, 0.0696)$
	$t = 1$	$(-0.01, 0.2434, 0.0603)$
3	$t = 0$	$(1.0445, 0, -0.6180)$
	$t = 0.2$	$(1.0601, 0.0661, -0.4765)$
	$t = 0.4$	$(1.4503, 0.2787, -0.7392)$
	$t = 0.6$	$(-0.5522, -1.6247, 1.3354)$
	$t = 0.8$	$(-0.1171, -0.4253, 0.3858)$
	$t = 1$	$(-0.1175, -0.3281, 0.3189)$
4	$t = 0$	$(1.0445, 0.4215, 1.6180)$
	$t = 0.2$	$(0.3667, 0.0869, 1.0054)$
	$t = 0.4$	$(0.8945, 0.2516, -0.3505)$
	$t = 0.6$	$(0.1901, 0.0218, 0.1309)$
	$t = 0.8$	$(0.1945, 0.0238, -0.006)$
	$t = 1$	$(0.4354, -0.5188, 0.0334)$
5	$t = 0$	$(-1.0445, 0.4215, 0)$

	$t = 0.2$	$(-0.9987, 0.2629, 0.6959)$
	$t = 0.4$	$(-1.7664, 0.0565, 1.1501)$
	$t = 0.6$	$(2.5364, 0.1411, -0.9545)$
	$t = 0.8$	$(0.2247, 0.0154, -0.0126)$
	$t = 1$	$(1.6071e-009, -9.7068e-010, -4.7904e-010)$
6	$t = 0$	$(0, 0.4215, -0.6180)$
	$t = 0.2$	$(0.2686, 0.4971, -0.7628)$
	$t = 0.4$	$(-0.0913, 0.1898, 0.2002)$
	$t = 0.6$	$(-0.0154, 0.1809, 0.0561)$
	$t = 0.8$	$(-0.0054, 0.1930, 0.0463)$
	$t = 1$	$(-0.01, 0.2434, 0.0603)$
7	$t = 0$	$(-1.0445, 0, 1.6180)$
	$t = 0.2$	$(1.2111, 0.6472, -2.8080)$
	$t = 0.4$	$(0.0211, -0.7467, 0.6008)$
	$t = 0.6$	$(1.7899e-009, 5.1575e-008, -6.4114e-009)$
	$t = 0.8$	$(-1.7182e-014, -4.9466e-013, 6.1488e-014)$
	$t = 1$	$(1.6478e-019, 4.7439e-018, -5.8968e-019)$
8	$t = 0$	$(-1.0445, -0.2965, -0.6180)$
	$t = 0.2$	$(-0.5604, -0.1354, -0.1705)$
	$t = 0.4$	$(0.0211, 0.0586, -0.1574)$
	$t = 0.6$	$(0.0407, -0.0223, 0.0749)$
	$t = 0.8$	$(0.2247, 0.0154, -0.0126)$
	$t = 1$	$(1.6071e-009, -9.7068e-010, -4.7904e-010)$
9	$t = 0$	$(0, 0.4215, 1.6180)$
	$t = 0.2$	$(0.7896, 0.8407, -1.9060)$
	$t = 0.4$	$(0.0246, 0.2664, -0.0579)$
	$t = 0.6$	$(-0.0316, -0.0523, 0.0960)$
	$t = 0.8$	$(-0.0307, -0.0489, 0.1240)$
	$t = 1$	$(-8.0191e-012, -1.3650e-010, 4.4157e-011)$
10	$t = 0$	$(0, -0.2965, 1.6180)$
	$t = 0.2$	$(0.1361, 0.0059, -0.4470)$
	$t = 0.4$	$(0.0108, -0.0328, -0.0251)$
	$t = 0.6$	$(-0.0029, 0.0318, 0.0102)$
	$t = 0.8$	$(-0.0025, 0.0436, 0.0117)$
	$t = 1$	$(2.7867e-007, -6.8887e-006, -1.7106e-006)$
11	$t = 0$	$(1.0445, -0.2965, 1.6180)$
	$t = 0.2$	$(0.2590, -0.7311, 1.4789)$
	$t = 0.4$	$(0.9774, 1.1004, -0.1782)$
	$t = 0.6$	$(0.5711, 0.2422, -0.0313)$
	$t = 0.8$	$(0.5873, -0.0262, 0.0024)$
	$t = 1$	$(0.7953, -0.1382, 0.0103)$
12	$t = 0$	$(-1.0445, 0.4215, 1.6180)$
	$t = 0.2$	$(1.0590, 0.0643, -0.4742)$
	$t = 0.4$	$(1.4503, 0.2787, -0.7392)$
	$t = 0.6$	$(0.0466, 1.3991, -0.1525)$
	$t = 0.8$	$(0.6414, -0.7668, -0.0242)$
	$t = 1$	$(-6.6616e-006, 1.8794e-005, -1.2334e-006)$

13	$t = 0$	$(-1.0445, -0.2965, 0)$
	$t = 0.2$	$(-0.5547, -0.1247, -0.1740)$
	$t = 0.4$	$(0.0211, 0.0586, -0.1574)$
	$t = 0.6$	$(0.0407, -0.0223, 0.0749)$
	$t = 0.8$	$(0.2247, 0.0154, -0.0126)$
	$t = 1$	$(1.6071e-009, -9.7068e-010, -4.7904e-010)$
14	$t = 0$	$(-1.0445, 0, -0.6180)$
	$t = 0.2$	$(-0.6285, -0.0182, -0.0056)$
	$t = 0.4$	$(-0.2331, -0.0287, 0.0046)$
	$t = 0.6$	$(0.18, -0.0245, 0.0028)$
	$t = 0.8$	$(0.2021, 0.0219, -0.0016)$
	$t = 1$	$(2.4234e-011, -2.4808e-011, 1.5681e-012)$
15	$t = 0$	$(1.0445, -0.2965, -0.6180)$
	$t = 0.2$	$(1.0601, 0.0649, -0.4765)$
	$t = 0.4$	$(1.4503, 0.2787, -0.7392)$
	$t = 0.6$	$(0.5505, -0.6183, 0.0679)$
	$t = 0.8$	$(0.2245, 0.0155, -0.0125)$
	$t = 1$	$(1.4030e-004, -8.4551e-005, -4.1859e-005)$
16	$t = 0$	$(1.0445, 0, 1.6180)$
	$t = 0.2$	$(0.3367, 0.0869, 1.0054)$
	$t = 0.4$	$(0.6479, 0.5745, -0.5514)$
	$t = 0.6$	$(0.0406, -0.0224, 0.0749)$
	$t = 0.8$	$(0.2247, 0.0154, -0.0126)$
	$t = 1$	$(1.6071e-009, -9.7068e-010, -4.7904e-010)$
17	$t = 0$	$(-1.0445, 0.4215, -0.6180)$
	$t = 0.2$	$(-0.2494, 0.4344, -0.3860)$
	$t = 0.4$	$(-0.2407, -0.0364, 0.0067)$
	$t = 0.6$	$(0.1799, -0.0246, 0.0029)$
	$t = 0.8$	$(0.2021, 0.0219, -0.0016)$
	$t = 1$	$(2.4102e-011, -2.4649e-011, 1.5569e-012)$
18	$t = 0$	$(-1.0445, -0.2965, 1.6180)$
	$t = 0.2$	$(2.0348e-011, 1.4921e-010, -1.6645e-011)$
	$t = 0.4$	$(-1.9262e-016, -1.6092e-015, 2.1414e-016)$
	$t = 0.6$	$(2.9907e-017, -6.3875e-017, 1.6956e-017)$
	$t = 0.8$	$(4.6647e-018, 2.2404e-017, -6.6030e-019)$
	$t = 1$	$(-4.4735e-023, -2.1486e-022, 6.3324e-024)$
19	$t = 0$	$(1.0445, 0.4215, -0.6180)$
	$t = 0.2$	$(1.0648, 0.3780, -0.5325)$
	$t = 0.4$	$(0.0125, 0.3051, -0.0298)$
	$t = 0.6$	$(-0.0316, -0.0523, 0.0960)$
	$t = 0.8$	$(-0.0307, -0.0489, 0.1240)$
	$t = 1$	$(-8.0423e-012, -1.3690e-010, 4.4284e-011)$
20	$t = 0$	$(0, -0.2965, -0.6180)$
	$t = 0.2$	$(0.1361, 0.0059, -0.4469)$
	$t = 0.4$	$(0.0108, -0.0328, -0.0251)$
	$t = 0.6$	$(-0.0029, 0.0318, 0.0102)$
	$t = 0.8$	$(-0.0025, 0.0436, 0.0117)$

	$t = 1$	$(2.7867e - 007, -6.8887e - 006, -1.7106e - 006)$
21	$t = 0$	$(-1.0445, 0.0001, 0.0001)$
	$t = 0.2$	$(-1.9041e - 005, -1.8310e - 004, 2.2813e - 005)$
	$t = 0.4$	$(1.1447e - 009, -1.4190e - 009, 2.5278e - 009)$
	$t = 0.6$	$(-5.9044e - 009, 1.1379e - 008, 9.0890e - 009)$
	$t = 0.8$	$(-2.1755e - 009, 6.3448e - 009, 3.2228e - 009)$
	$t = 1$	$(-9.3777e - 012, 3.3079e - 011, 1.3616e - 011)$
22	$t = 0$	$(1.0445, 0.0001, 0.0001)$
	$t = 0.2$	$(1.0648, 0.3782, -0.5325)$
	$t = 0.4$	$(0.6479, 0.5745, -0.5514)$
	$t = 0.6$	$(1.0406, -0.0224, 0.0749)$
	$t = 0.8$	$(-0.0371, -0.2601, 0.2927)$
	$t = 1$	$(-0.0360, -0.2543, 0.2875)$
23	$t = 0$	$(0.0001, 0.4215, 0.0001)$
	$t = 0.2$	$(0.0378, 0.3639, -0.0452)$
	$t = 0.4$	$(-0.0912, 0.1898, 0.2002)$
	$t = 0.6$	$(-0.0154, 0.1809, 0.0561)$
	$t = 0.8$	$(-0.0053, 0.1930, 0.0463)$
	$t = 1$	$(-0.01, 0.2434, 0.0603)$
24	$t = 0$	$(0.0001, -0.2965, 0.0001)$
	$t = 0.2$	$(-0.0195, -0.1868, 0.0234)$
	$t = 0.4$	$(0.0108, -0.0328, -0.0251)$
	$t = 0.6$	$(-0.0029, 0.0318, 0.0102)$
	$t = 0.8$	$(-0.0025, 0.0436, 0.0117)$
	$t = 1$	$(2.7867e - 007, -6.8887e - 006, -1.7106e - 006)$
25	$t = 0$	$(0.0001, 0.0001, 1.6180)$
	$t = 0.2$	$(-1.1467, -0.3255, 3.6821)$
	$t = 0.4$	$(-0.0913, 0.1898, 0.2002)$
	$t = 0.6$	$(-0.0154, 0.1809, 0.0561)$
	$t = 0.8$	$(-0.0053, 0.1930, 0.0463)$
	$t = 1$	$(-0.01, 0.2434, 0.0603)$
26	$t = 0$	$(0.0001, 0.0001, -0.6180)$
	$t = 0.2$	$(-0.0189, -0.1810, 0.0227)$
	$t = 0.4$	$(0.0108, -0.0328, -0.0251)$
	$t = 0.6$	$(-0.0029, 0.0318, 0.0102)$
	$t = 0.8$	$(-0.0025, 0.0436, 0.0117)$
	$t = 1$	$(2.7867e - 007, -6.8887e - 006, -1.7106e - 006)$
27	$t = 0$	$(0.0001, 0.0001, 0.0001)$
	$t = 0.2$	$(9.5902e - 010, -9.5902e - 010, -9.5902e - 010)$
	$t = 0.4$	$(9.1915e - 015, 9.2569e - 015, 9.1703e - 015)$
	$t = 0.6$	$(8.8977e - 018, -3.7991e - 018, -1.6723e - 017)$
	$t = 0.8$	$(-1.0151e - 018, -3.7991e - 018, 3.1026e - 018)$
	$t = 1$	$(9.7350e - 024, 3.6434e - 023, -2.9755e - 023)$

The following Table (1.4.2) gives the eigenvectors corresponding to the eigenvalues obtained from Table (1.4.1) of the problem (1.4.1)

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Table 1.4.2: Estimate of eigenvectors for different starting eigenvalues

Sl. No.	**Eigenvalue**$(\lambda_1, \lambda_2, \lambda_3)$	**Eigenvector**
1	$(2.7867e-007, -6.8887e-006, -1.7106e-006)$	$x = \begin{pmatrix} -0.0000 \\ -0.7173 \\ 0.6967 \end{pmatrix}$ $y = \begin{pmatrix} 0.8817 \\ 0.0000 \\ 0.4720 \end{pmatrix}$ $z = \begin{pmatrix} -0.6909 \\ 0.7230 \\ 0.0000 \end{pmatrix}$
2	$(-0.01, 0.2434, 0.0603)$	$x = \begin{pmatrix} -0.0547 \\ 0.7060 \\ -0.7061 \end{pmatrix}$ $y = \begin{pmatrix} 0.1946 \\ -0.9747 \\ 0.1096 \end{pmatrix}$ $z = \begin{pmatrix} -0.7547 \\ 0.6542 \\ 0.0496 \end{pmatrix}$
3	$(-0.1175, -0.3281, 0.3189)$	$x = \begin{pmatrix} -0.9842 \\ -0.0646 \\ 0.1651 \end{pmatrix}$ $y = \begin{pmatrix} 0.2590 \\ 0.9302 \\ -0.2601 \end{pmatrix}$ $z = \begin{pmatrix} 0.3910 \\ 0.9075 \\ -0.1538 \end{pmatrix}$
4	$(0.4354, -0.5188, 0.0334)$	$x = \begin{pmatrix} -0.9957 \\ -0.0664 \\ 0.0651 \end{pmatrix}$ $y = \begin{pmatrix} 0.7774 \\ 0.0032 \\ 0.6290 \end{pmatrix}$ $z = \begin{pmatrix} 0.7529 \\ 0.6045 \\ -0.2600 \end{pmatrix}$
5	$(1.6071e-009, -9.7068e-010, -4.7904e-010)$	$x = \begin{pmatrix} 0.0000 \\ -0.4506 \\ -0.8927 \end{pmatrix}$ $y = \begin{pmatrix} -0.0518 \\ 0.0000 \\ 0.9987 \end{pmatrix}$ $z = \begin{pmatrix} 0.8246 \\ 0.5657 \\ 0.0000 \end{pmatrix}$
6	$(1.6478e-019, 4.7439e-018, -5.8968e-019)$	$x = \begin{pmatrix} 0.000 \\ -0.7071 \\ 0.7071 \end{pmatrix}$ $y = \begin{pmatrix} 0.7071 \\ 0.0000 \\ -0.7071 \end{pmatrix}$ $z = \begin{pmatrix} 0.7071 \\ -0.7071 \\ 0.0000 \end{pmatrix}$
7	$(-8.0191e-012, -1.3650e-010, 4.4157-011)$	$x = \begin{pmatrix} 0.0000 \\ 0.6754 \\ 0.7374 \end{pmatrix}$ $y = \begin{pmatrix} 0.7268 \\ 0.0000 \\ 0.6868 \end{pmatrix}$

		$z = \begin{pmatrix} 0.4607 \\ -0.8876 \\ 0.0000 \end{pmatrix}$
8	$(0.7953, -0.1382, 0.0103)$	$x = \begin{pmatrix} 0.8301 \\ -0.3267 \\ -0.4520 \end{pmatrix}$ $y = \begin{pmatrix} -0.6741 \\ 0.0315 \\ 0.7380 \end{pmatrix}$ $z = \begin{pmatrix} -0.0118 \\ -0.2848 \\ 0.9585 \end{pmatrix}$
9	$(-6.6616e-006, 1.8794e-005, -1.2334e-006)$	$x = \begin{pmatrix} 0.0000 \\ -0.6630 \\ -0.7486 \end{pmatrix}$ $y = \begin{pmatrix} -0.7599 \\ 0.0000 \\ 0.6501 \end{pmatrix}$ $z = \begin{pmatrix} 0.7354 \\ -0.6782 \\ 0.0000 \end{pmatrix}$
10	$(2.4234e-011, -2.4808e-011, 1.5681e-012)$	$x = \begin{pmatrix} 0.0000 \\ -0.6484 \\ -0.7613 \end{pmatrix}$ $y = \begin{pmatrix} -0.8925 \\ 0.0000 \\ 0.4510 \end{pmatrix}$ $z = \begin{pmatrix} -0.8162 \\ -0.5777 \\ 0.0000 \end{pmatrix}$
11	$(1.4030e-004, -8.4551e-005, -4.1859e-005)$	$x = \begin{pmatrix} 0.0000 \\ -0.5793 \\ -0.8151 \end{pmatrix}$ $y = \begin{pmatrix} -0.3308 \\ 0.0000 \\ -0.9460 \end{pmatrix}$ $z = \begin{pmatrix} -0.7953 \\ -0.6063 \\ 0.0001 \end{pmatrix}$
12	$(2.4102e-011, -2.4649e-011, 1.5569e-012)$	$x = \begin{pmatrix} -0.0000 \\ -0.6486 \\ -0.7611 \end{pmatrix}$ $y = \begin{pmatrix} -0.8925 \\ -0.0000 \\ 0.4511 \end{pmatrix}$ $z = \begin{pmatrix} -0.8162 \\ -0.5778 \\ 0.0000 \end{pmatrix}$
13	$(-4.4735e-023, -2.1486e-022, 6.3324e-024)$	$x = \begin{pmatrix} -0.0000 \\ -0.7071 \\ 0.7071 \end{pmatrix}$ $y = \begin{pmatrix} 0.7071 \\ -0.0000 \\ -0.7071 \end{pmatrix}$ $z = \begin{pmatrix} -0.7071 \\ 0.7071 \\ -0.0000 \end{pmatrix}$
14	$(-8.0423e-012, -1.3690e-010, 4.4284e-011)$	$x = \begin{pmatrix} 0.0000 \\ -0.6755 \\ -0.7374 \end{pmatrix}$ $y = \begin{pmatrix} -0.7267 \\ 0.0000 \\ -0.6869 \end{pmatrix}$

		$z = \begin{pmatrix} -0.4606 \\ 0.8876 \\ 0.0000 \end{pmatrix}$
15	$(-9.3777e-012, 3.3079e-011, 1.3616e-011)$	$x = \begin{pmatrix} -0.0000 \\ 0.7029 \\ -0.7113 \end{pmatrix}$ $y = \begin{pmatrix} -0.7090 \\ -0.0000 \\ 0.7057 \end{pmatrix}$ $z = \begin{pmatrix} -0.7070 \\ 0.7070 \\ 0.0000 \end{pmatrix}$
16	$(-0.0360, -0.2543, 0.2875)$	$x = \begin{pmatrix} 0.9634 \\ -0.0449 \\ -0.2643 \end{pmatrix}$ $y = \begin{pmatrix} -0.7095 \\ 0.1037 \\ -0.6972 \end{pmatrix}$ $z = \begin{pmatrix} -0.9537 \\ 0.2314 \\ 0.1920 \end{pmatrix}$
17	$(9.7350e-024, 3.6434e-023, -2.9755e-023)$	$x = \begin{pmatrix} 0.0000 \\ -0.7071 \\ 0.7071 \end{pmatrix}$ $y = \begin{pmatrix} -0.7071 \\ 0.0000 \\ 0.7071 \end{pmatrix}$ $z = \begin{pmatrix} 0.0000 \\ -0.7071 \\ 0.7071 \end{pmatrix}$

V. COCLUSION

Since the successive differences between the computed eigenvalues tend to zero, the method converges rapidly to the exact eigenvalue. Thus, it has been demonstrated that the method is convergent, and it can therefore be extended to solve three-parameter or higher-order eigenvalue problems. Another advantage of this approach is that it does not require an initial guess, as the eigenvalues corresponding to $t = 0$ serve as the initial approximations. Furthermore, the method yields both eigenvalues and eigenvectors simultaneously.

However, if the solution path passes through a singular point, switching between homotopy curves may occur. To prevent such switching in the case of two-parameter right-definite eigenvalue problems, Plestenjak [14] derived a bound η that is independent of t. The present work does not discuss that procedure in detail. Instead, by appropriately selecting W_1, W_2 and W_3 at $t = 0$, the singularity at $t = 0$ is avoided.

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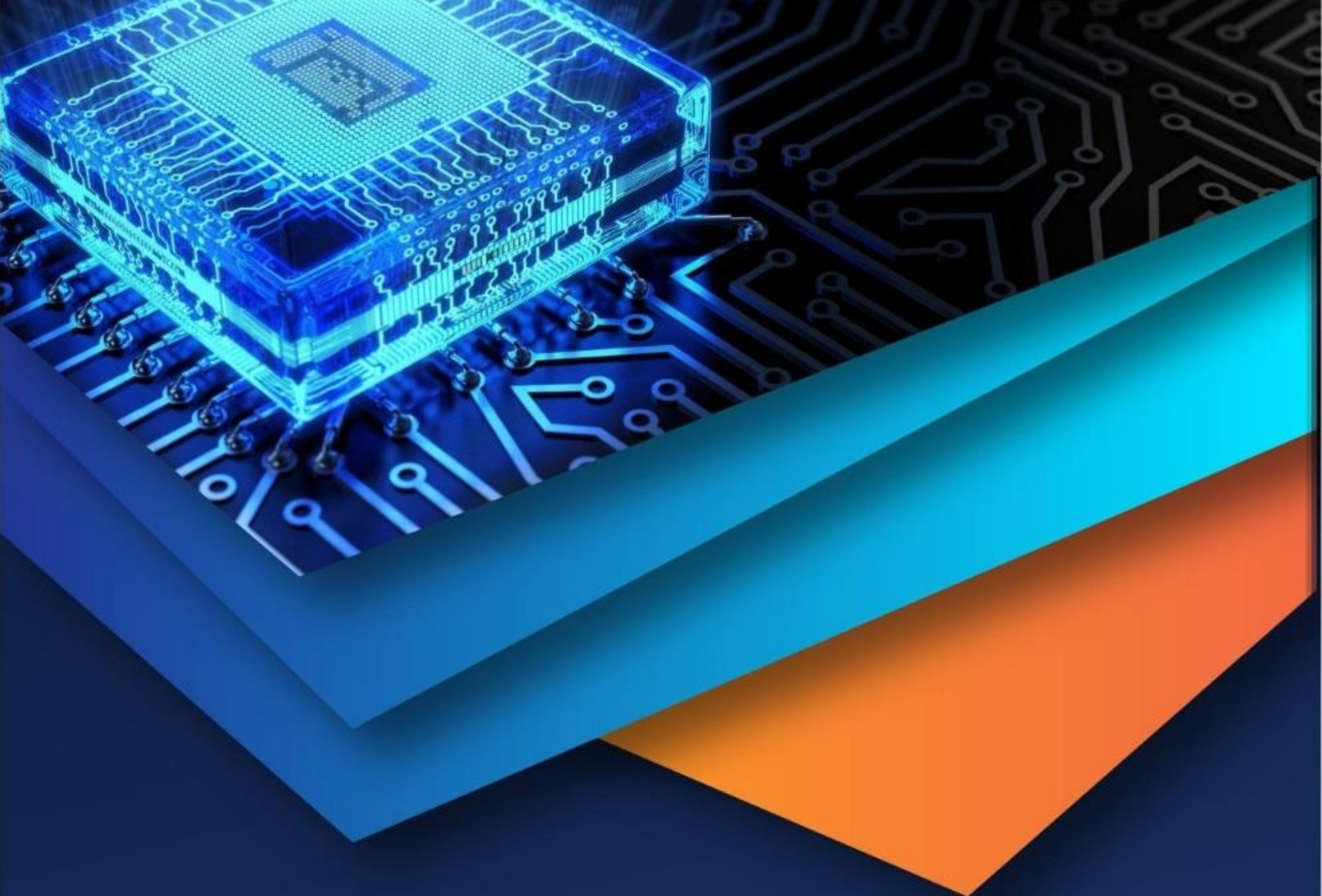

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