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CPT Invariance of Quaternion Dirac equation

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Abstract: In this paper the invariance of Quaternion Dirac equation under Lorentz Transformation, Charge conjugation, Parity transformation and Time reversal operation has been discussed successfully. The invariance under the combined operation of Charge conjugation, Parity and Time reversal (CPT) has also been discussed and expression for C, P, T and combined CPT operators have been obtained in terms of quaternions. Invariance condition for electric and magnetic field has also been obtained. It has been concluded that the Quaternion Dirac equation dominates over ordinary Dirac equation because of the advantage of algebra of quaternions.

I. INTRODUCTION

Quaternions have the same properties as complex numbers but differs in the way that commutative law is not valid for quaternions, which has many advantages over complex numbers. They were first invented by Sir W.R. Hamilton [1] and several authors worked on quaternions [2-5]. Quaternions are the example of hypercomplex numbers. Quaternions means set of four and introduces new methods in Physics and Mathematics. Quaternions are important mathematical tools which are very useful in construction of four-dimensional world. Quaternion Quantum Mechanics is a new type of Mechanics having inner products, matrix elements and Quaternion coefficients. Adler [6-8] gave the idea that Quaternion Quantum Mechanics offers elegant substructure well for quarks and leptons.

Relativistic Quantum Mechanics is one of the important theories of Quantum Mechanics. Since Relativistic Quantum Mechanics includes 3+1 space-time dimension and it becomes difficult to explain since space and time are different coordinates. If we use Quaternions, then they have unique advantage that space and time are treated equal and can be defined by four component Quaternionic function. Quaternion provides four-dimensional structure to relativistic quantum mechanics, also Quaternion structure is compact and theory becomes simplified. Quaternions in matrix form can also be represented in terms of Pauli spin matrices [9], so spin is natural outcome of Quaternion Dirac equation, while in ordinary Dirac equation spin has to introduce by hand. The main contribution in Quaternion Quantum Mechanics was made by Finkelstein [10] A.J. Davies [11], Rotelli [12], Leo et. al. [13-14] and Rawat et. al. [16-19]. Most important use of Quaternion is in Relativistic Quantum Mechanics.

Lorentz invariance and CPT invariance are two of the most fundamental symmetry of nature. For any equation to be successful in physics, it is necessary that it should remain invariant in all frames of references. Although violation of individual C, P, T has been observed in some interaction but the violation of combined CPT has never been observed so far. CPT theorems were introduced by Luders and Pauli [20]. Quaternion Dirac equation should also be tested for invariance under proper Lorentz transformation, Charge conjugation, Parity and Time reversal operations.

In this paper the invariance of Quaternion Dirac equation under Lorentz Transformation, Charge conjugation, Parity transformation and Time reversal operation has been discussed separately. The invariance under the combined operation of Charge conjugation, Parity and Time reversal (CPT) has also been discussed. It has been concluded that the Quaternion Dirac equation dominates over ordinary Dirac equation because of the advantage of algebra of quaternions.

II. DEFINITIONS

Quaternionic function M is defined as

$$M = M_0 + e_1 M_1 + e_2 M_2 + e_3 M_3 \qquad (1)$$

Equation (1) is made up of real and imaginary part, where $e_1 M_1 + e_2 M_2 + e_3 M_3$ is imaginary part while M_0 is real part and $M_0, M_1, M_2 \& M_3$ are real and e_1, e_2 and e_3 are quaternion units, which satisfy following properties.

$$e_1{}^2 = e_2{}^2 = e_3{}^2 = -1 \implies e_j{}^2 = -1$$
$$e_1 e_2 = -e_2 e_1 = e_3$$
$$e_2 e_3 = -e_3 e_2 = e_1$$
$$e_3 e_1 = -e_1 e_3 = e_2 \ or \ e_i e_j = \delta_{ij} + \varepsilon_{ijk} e_k$$
$$\forall \ i, j, k = 1,2,3 \qquad (2)$$

Where δ_{ij} is kronecker delta and ε_{ijk} is three index Levi Civita symbol with $\varepsilon_{ijk} = +1$

Quaternionic conjugate of equation (1) is defined as

$$\bar{M} = M_0 - e_1 M_1 - e_2 M_2 - e_3 M_3 \tag{3}$$

Instead of writing in four components, Quaternions can be written in terms of two components, known as symplectic representation

$$M = M_\alpha + e_2 M_\beta$$

where $M_\alpha = M_0 + e_1 M_1$ and $M_\beta = M_2 - e_1 M_3$ $\tag{4}$

Complex conjugate of M_α & M_β can be written as

$$\overline{M_\alpha} = M_0 - e_1 M_1$$
$$\overline{M_\beta} = M_2 + e_1 M_3 \tag{5}$$

Also $\bar{\bar{M}} = M$

Norm of quaternion is defined as

$$N(M) = (\bar{M}M)^{\frac{1}{2}} = (M\bar{M})^{\frac{1}{2}} = (M_0^2 + M_1^2 + M_2^2 + M_3^2)^{\frac{1}{2}} \tag{6}$$

Also $N(M_1 M_2) = N(M_2)N(M_1)$ $\tag{7}$

Inverse of Quaternion is defined by

$$M^{-1} = \frac{\bar{M}}{|M|^2} \tag{8}$$

Quaternion conjugate satisfies the following properties

$$\overline{(M_1 M_2)} = \overline{M_2} . \overline{M_1} \tag{9}$$

Now sum and product of Quaternions are defined as

$$(M_0, M) + (N_0, N) = (M_0 + N_0, M + N) \tag{10}$$
$$(M_0, M).(N_0, N) = (M_0 N_0 - M.N, M_0 N + M.N) \tag{11}$$

Quaternion elements are non-abelian in nature and thus represents a division ring. In terms of Pauli's spin matrices Quaternions in matrix form are represented by

Since

$$e_l = -i\sigma_l \tag{12}$$

$$e_1 = -i\sigma_1 = -i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$e_2 = -i\sigma_2 = -i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$e_3 = -i\sigma_3 = -i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \tag{13}$$

III.   INVARIANCE OF QUATERNION DIRAC EQUATION UNDER PROPER LORENTZ TRANSFORMATION

Lorentz covariance of an equation means it has same form in all frame of reference and can be built up from succession of infinitesimal transformations. To check the consistency of Quaternion Dirac equation with theory of relativity, we will check its invariance under Lorentz transformation. The four-space component is defined as

$$\{x_\mu\} = (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, ict)$$
$$= (-x_0, x_1, x_2, x_3) = (-t, x_1, x_2, x_3) \tag{14}$$

Where x_μ is quaternion function and we have taken natural units here (c = ℏ =1).

Homogeneous Lorentz transformations are given as

$$x_\mu = x'_\mu = \alpha_\mu^\nu x'_\nu \text{ or } \qquad x' = \alpha x \tag{15}$$

Hence we can establish the relation between ψ(x) and ψ'(x)

As $\psi'(x) = S\psi(x)$ $\tag{16}$

Where S is 4×4 matrix

From equation (14) and (15)

$$\partial x_\mu = \alpha_\mu^\nu \partial x'_\nu \qquad \text{or} \qquad \partial_\mu = \alpha_\mu^\nu \partial'_\nu$$

And Quaternionic Dirac equation becomes

$S\left(\sum_{\mu=1}^{4} i\Upsilon_{\mu}\alpha_{\mu}^{\nu}\partial'_{\nu} - m\right)S^{-1}\psi'(x,t) = 0$

Or

$\left(\sum_{\mu=1}^{4} Si\Upsilon_{\mu}S^{-1}\alpha_{\mu}^{\nu}\partial'_{\nu} - SmS^{-1}\right)\psi'(x,t) = 0$ (17)

Let us introduce Quaternionic potentials $A\mu$ and $B\mu$ defined as electric and magnetic field

$A_{\mu} = A_0 + e_1A_1 + e_2A_2 + e_3A_3 = A_0 + \sum_{l=1}^{3} e_lA_l$

$B_{\mu} = A_0 + e_1B_1 + e_2B_2 + e_3B_3 = B_0 + \sum_{l=1}^{3} e_lB_l$ (18)

In electromagnetic field for two gauge potentials $A\mu$ and $B\mu$,Invariance of Dirac equation can be defined as-[17]

$S\{\Upsilon\mu\,(i\partial\mu\,\psi + ieA\mu\,\psi - ie\psi B\mu) + m\psi\}S^{-1}{=}0$

$S\{\Upsilon\mu\,(i\partial\mu\,\psi + ieA\mu\,\psi - ie\psi B\mu) + m\psi\}S^{-1}{=}0$

$\{\Upsilon\mu\,(i\partial\mu\,\psi' + ieA\mu\,\psi' - ie\psi'B\mu) + m\psi'\}{=}0$ (18)

Where $\Upsilon\mu$ are Dirac's Υ matrices defined as follows

$\Upsilon_0 = \beta, \; \Upsilon_l = \beta\alpha_l$

$\Upsilon_0 = \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $, \Upsilon_l = \begin{bmatrix} 0 & ie_l \\ -ie_l & 0 \end{bmatrix}$

e_l are quaternion matrices

For the invariance of Dirac equation under Lorentz transformation

$S\Upsilon_{\mu}S^{-1} = \Upsilon_{\mu}$

$SA_{\mu}S^{-1} = A_{\mu}$

$SB_{\mu}S^{-1} = B_{\mu}$ (19)

Where S and S⁻¹ are defined by

$S = 1 + \frac{1}{8}(\Upsilon_{\mu},\Upsilon_{\nu})\varepsilon_{\mu\nu}$

$S^{-1} = 1 - \frac{1}{8}(\Upsilon_{\mu},\Upsilon_{\nu})\varepsilon_{\mu\nu}$ (20)

Here we can see that S is defined in terms of Dirac matrices, so S corresponds to infinitesimal transformations and leads to proper Lorentz transformation

A. *Invariance of Quaternion Dirac Equation under Charge Conjugation*

Charge conjugation operator changes a particle into its antiparticle or vice versa. According to hole theory absence of charge e with negative energy is equivalent to presence of positive energy operation, charge conjugation operation is

$\psi' = \psi_c = C\psi^{\dagger}$ (21)

and

$GM = C\bar{M}$ (22)

where $\bar{M}$ is quaternion conjugate and C is 4×4 matrix. Using equation (22) and equation (18) we get

$G\{\Upsilon\mu\,(i\partial\mu\,\psi + ieA\mu\,\psi - ie\psi B\mu) + m\psi\} = 0$

$C\{\Upsilon\mu\,(i\partial\mu\,\psi + ieA\mu\,\psi - ie\psi B\mu) + m\psi\}^{\dagger} = 0$ (23)

As such equation (23) can be retained from equation (22) if

$C\Upsilon_l^{\dagger}C^{-1} = \Upsilon_l$

$C\Upsilon_0^{\dagger}C^{-1} = \Upsilon_0$

For electric field

$CA_l^{\dagger}C^{-1} = A_l$

$CA_0^{\dagger}C^{-1} = A_0$ (24)

For Magnetic field

$CB_l^{\dagger}C^{-1} = B_l$

$CB_0^{\dagger}C^{-1} = B_0$ (25)

$\Upsilon_1 \; and \; \Upsilon_3$ are imaginary

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$$CY_1^\dagger C^{-1} = -Y_1,$$
$$CY_3^\dagger C^{-1} = -Y_3,$$
$$CY_0^\dagger C^{-1} = -Y_0$$
And
$$CY_2^\dagger C^{-1} = Y_2 \tag{26}$$
Since $Y_l = \beta\alpha_l$, so that

$$Y_0 = \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$Y_1 = \beta\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

$$Y_2 = \beta\alpha_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$Y_3 = \beta\alpha_3 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \tag{27}$$

And Charge conjugation operation is defined as

$$C = Y_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

where $C^2 = 1$, shows that C is unitary matrix.

B. Invariance of Quaternion Dirac Equation under Parity operation

Under Parity operation only space coordinate changes sign but time coordinate does not change sign
$$\vec{x'} = -\vec{x} \text{ and } t' = t \quad \text{ or } (ict)' = -ict \text{ i.e. } \vec{x_0'} = -\vec{x_0'} \tag{29}$$ These transformations are known as parity operations which change right-handed coordinate system to a left-handed system.

Parity transformation is defined as
$$\psi = \psi(-x, t) = P(x, t)$$
Where P is the Parity operator, which requires
$$P^{-1}Y_l P = -Y_l \text{ and } P^{-1}Y_0 P = -Y_0 \tag{30}$$
For electric field
$$PA_l^\dagger P^{-1} = -A_l$$
$$PA_0^\dagger P^{-1} = A_0 \tag{31}$$
For Magnetic field
$$PB_l^\dagger P^{-1} = -B_l$$
$$PA_0^\dagger P^{-1} = B_0 \tag{32}$$

where

$$P = Y_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{33}$$ which satisfies equation (28).

Here $P^2 = 1$ *or* $P = \pm 1$
Which suggests that Parity operator has two eigen values. P = +1 suggests positive energy states and P = -1 suggests negative energy states which represents positron state and shows that e^+ and e^- have opposite parity states.

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C. Invariance of Quaternion Dirac Equation under Time Reversal

Time reversal operator changes sign of time only and leaves space coordinate unchanged. Invariance under time reversal is defined as

$\vec{x'} = \vec{x}$ and $t' = -t$

And wave function becomes

$$\psi'(t') = \Gamma\psi(t) \tag{34}$$

So that equation $H\psi = i\frac{\partial\psi}{\partial t}$ using equation (34) becomes

$$\Gamma H\Gamma^{-1}\psi'(t') = \Gamma i\Gamma^{-1}\frac{\partial\psi'}{\partial t'}$$

For the invariance of this equation, we have to take into account

$$\Gamma H\Gamma^{-1} = H \ , \ \ \Gamma i\Gamma^{-1} = -i \tag{35}$$

and

$$\psi'(t') = T\,\psi^\dagger(t) \tag{36}$$

Γ and T are related by

$$\Gamma M = T\bar{M} \tag{37}$$ Where T is 4×4 matrix

and $\bar{M}$ is quaternion conjugate. The Dirac equation in electromagnetic field is given by

$$\Upsilon_\mu\left(i\,\partial_\mu\psi + ie\mathrm{A}_\mu\psi - ie\psi\mathrm{B}_\mu\right) + m\psi = 0 \tag{38}$$

Which by operating with Γ becomes

$$\Gamma\{\Upsilon\mu\,(i\partial\mu\,\psi + ieA\mu\,\psi - ie\psi B\mu) + m\psi\} = 0 \tag{39}$$

So that

$$T\{\Upsilon\mu\,(i\partial\mu\,\psi + ieA\mu\,\psi - ie\psi B\mu) + m\psi\}^\dagger = 0 \tag{40}$$

By explicit calculations it can be shown that equation (40) retains equation (38) only if

$$T\Upsilon_l^\dagger T^{-1} = \Upsilon_l \qquad \text{and} \quad T\Upsilon_0^\dagger T^{-1} = \Upsilon_0 \tag{41}$$

For electric field

$$T A_l^\dagger T^{-1} = A_l$$
$$T A_0^\dagger T^{-1} = A_0 \tag{42}$$

For Magnetic field

$$T B_l^\dagger T^{-1} = B_l$$
$$T B_0^\dagger T^{-1} = B_0 \tag{43}$$

where $\Upsilon_0, \Upsilon_1, \Upsilon_2 \& \Upsilon_3$ are defined by equation (27)

Here $\Upsilon_1 \& \Upsilon_3$ *are imaginary while* $\Upsilon_0 \& \Upsilon_2$ *are real*

$$T\Upsilon_1^\dagger T^{-1} = -\Upsilon_1,$$
$$T\Upsilon_2^\dagger T^{-1} = \Upsilon_2,$$
$$T\Upsilon_3^\dagger T^{-1} = -\Upsilon_3$$
$$T\Upsilon_0^\dagger T^{-1} = \Upsilon_0 \tag{44}$$

These relations (39) are satisfied only when we take

$$T = \Upsilon_1\Upsilon_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = e_2 I \tag{45}$$

IV.   INVARIANCE OF QUATERNION DIRAC EQUATION UNDER COMBINED CPT OPERATION

Combined CPT operator is defined as combined operation of charge, parity and time reversal operator, which is given as follow

$$\psi(x') = PCT\psi(x) = PCT\psi$$

$$= PC\Upsilon_1\Upsilon_3\psi^\dagger(x) = P\Upsilon_2\Upsilon_1\Upsilon_3\psi(x) = \Upsilon_0\Upsilon_2\Upsilon_1\Upsilon_3\psi(x) = \Upsilon_5\psi(x) \tag{46}$$

From equations (25)

$$PCT = \Upsilon_5 = \Upsilon_0\Upsilon_2\Upsilon_1\Upsilon_3$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{47}$$

Positron wave function $= \Upsilon_5 \times$ electron wave function. $\qquad\qquad$ (48)

V. DISCUSSION

Quaternion Dirac equation in electro-magnetic field must remain invariant in all frame of references and it has been observed that Dirac equation remains invariant under Proper Lorentz Transformation, Parity transformation, Time reversal, Charge conjugation and under combined operation of Charge conjugation, Parity transformation and Time reversal (CPT) in the same manner as for the case of usual Dirac equation. Expressions for C,P,T and CPT have been obtained in terms of quaternions. Invariance condition for electric and magnetic field has also been obtained. It has been observed that Quaternionic Dirac equation consists of all the transformations as usual Dirac equation does, so we have developed Quaternion formalism of Dirac equation in simple, compact and consistent manner because of the advance algebraic structure of Quaternions and spin is natural outcome of quaternionic components.

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