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Enduring Sequences of Hyperbolic Diophantine Quadruples involving Hilbert and Gnomonic Number

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Abstract: This paper introduces Hyperbolic Diophantine quadruple, defined as set of four distinct positive integers satisfying the property $D(n)$, where pairwise product plus n yields perfect square. A novel construction generates sequences of such quadruples by first forming a Hyperbolic pair using Hilbert number and Gnomonic number. Algebraic manipulations extend this pair to a triple, then to a quadruple via a conjecture ensuring the fourth element preserves $D(n)$.
Keywords: Diophantine quadruple, Hyperbolic number, Hilbert number, Gnomonic number, Integer sequence.
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I. INTRODUCTION

Diophantine q-tuple with the property $D(n)$ represent a significant extension of the classical problem posed by Diophantus, who sought set of positive rational numbers where the product of any two distinct elements plus a fixed n yields perfect square. Formally, a set $(a_1, a_2, \dots \dots \dots . . a_q)$ of q distinct positive integers possess the property $D(n)$, for fixed n, if $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq q$. While Diophantus identified a quadruple in rational $(\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16})$ for $n = 1$, the integer case has proven more challenging, sparking extensive research into their structure and generation [1-3,9,10]. Constructions of such q-tuple involving polygonal and pyramidal numbers, which feature prominently in generating triples and quadruples due to their intriguing properties often rely on solutions to Pell equations and Continued fraction expansions [5-8,11-14].

This work pioneers Hyperbolic Diophantine pairs, novel constructs involving Hilbert number and Gnomonic number then extend these pairs to triples preserving the $D(n)$ condition across all pairs. Conjecture 1 in [4] asserts that every such triple admits a unique fourth tuple, forming a Hyperbolic Diophantine quadruple. Recursive application yields infinite sequence.

II. PRELIMINARIES

1) *Hilbert number:* Hilbert numbers, named after the renowned mathematician David Hilbert, are a specific set of numbers that include all positive integers of the form $4k + 1, k \geq 0$.
2) *Gnomonic number:* A figurate number of the form $g_k = 2k - 1$, giving the area of the square gnomon obtained by removing a square of side $k - 1$ from a square of side k.
3) *Hyperbolic number:* A Hyperbolic number is an ordered pair of real numbers $P = p_1 + p_2 j$, where p_1, p_2 are real numbers and j is the Hyperbolic unit satisfying $j^2 = 1$ and $j \neq \pm 1$.

III. SEQUENCE CONSTRUCTION

Consider distinct Hyperbolic numbers denoted as

$a = \mathscr{H}_k + 3 + 3yj = 4k + 4 + 3yj, \quad b = \mathscr{H}_{k+1} + 3 + 3yj = 4k + 8 + 3yj$

Their product plus four yields a perfect square that is,

$$ab + 4 = \alpha^2 \qquad (3.1)$$

This establishes a Diophantine pair $(4k + 4 + 3yj, 4k + 8 + 3yj)$ with the property $D(n)$ for $n = 4$.

Similarly, consider

$r = g_{k+1} + 3 + 3yj = 2k + 4 + 3yj, \quad s = g_{k+2} + 3 + 3yj = 2k + 6 + 3yj$

such that $(2k + 4 + 3yj, 2k + 6 + 3yj)$ forms a Diophantine pair for $n = 1$ satisfying,

$$rs + 1 = \mu^2 \qquad (3.2)$$

Algebraic techniques detailed in subsequent subsections systematically extend these pairs through triples to quadruples, thereby generating infinite sequences while preserving the $D(n)$ throughout the construction process.

A. Sequence Construction of Hyperbolic Diophantine Quadruple Involving Hilbert number:

Having established the initial Hyperbolic Diophantine pair $(4k + 4 + 3yj, 4k + 8 + 3yj)$, this section is to form sequence of triples and extending it to quadruples that preserves the property $D(4)$ across all pairs.

For the pair $(4k + 4 + 3yj, 4k + 8 + 3yj)$, let $c$serve as a third tuple (positive integer)

$$ac + 4 = \beta^2 \qquad (3.1.1)$$

$$bc + 4 = \gamma^2 \qquad (3.1.2)$$

Assume $\beta = a + \alpha$ and $\gamma = b + \alpha$. Substituting these linear conversions in (3.1.1) and (3.1.2) and deducting leads to

$$(a - b)c = (a + \alpha)^2 - (b + \alpha)^2 \qquad (3.1.3)$$

Upon commutating, the value of c is identified as,

$$c = 16k + 24 + 12yj \qquad (3.1.4)$$

(a, b, c) is a Hyperbolic Diophantine triple with $D(4)$.

The fourth tuple is subsequently obtained by applying the conjecture 1 [4].

$$d = 81y^3j^3 + 324ky^2j^2 + 477y^2j^2 + 432k^2 yj + 1272kyj + 918yj + 192k^3 + 848k^2 + 1224k + 580$$

Thus, (a, b, c, d) is a Hyperbolic Diophantine quadruple with $D(4)$.

To construct sequence, the pair (b, c) now serves as the basis for progression.

For the pair (b, c), let e be the third tuple such that,

$$be + 4 = \beta^2 \qquad (3.1.5)$$

$$ce + 4 = \gamma^2 \qquad (3.1.6)$$

Applying linear transformation $\beta = b + \alpha$; $\gamma = c + \alpha$ in (3.1.5), (3.1.6) and performing the algebraic algorithms yields

$$e = 36k + 60 + 27yj \qquad (3.1.7)$$

Forming a Hyperbolic Diophantine triple (b, c, e)

The fourth tuple f is then obtained as

$$f = 810y^3j^3 + 3240ky^2j^2 + 5544y^2j^2 + 4320k^2yj + 14784kyj + 12594yj + 1920k^3 + 9856k^2 + 16792k + 9500$$

Completing the quadruple (b, c, e, f) with $D(4)$.

The pair (c, e) now becomes the new basis. The third tuple l satisfies

$$cl + 4 = \beta^2 \qquad (3.1.8)$$

$$el + 4 = \gamma^2 \qquad (3.1.9)$$

The corresponding third and fourth tuples are determined as,

$$l = 100k + 160 + 75yj$$

$$m = 24300y^3j^3 + 97200ky^2j^2 + 154440k^2j^2 + 129600k^2yj + 411840kyj + 327108yj + 57600k^3 + 274560k^2 + 436144k + 230888$$

(c, e, l, m) is a $D(4)$ Hyperbolic Diophantine quadruple.

This recursive process continues indefinitely.

Through this recurring procedure, the sequence of Hyperbolic Diophantine quadruple with $D(4)$ involving Hilbert number emerges $\{(a, b, c, d), (b, c, e, f), (c, e, l, m) \ldots\ldots, \ldots.\}$

Few sequences are tabulated below for $y = 1$ and varying k

k	y	$(\{a, b, c, d\}\{b, c, e, f\}\{c, e, l, m\} \ldots\ldots\ldots, \ldots)$
1	1	$\{(8 + 3j, 12 + 3j, 40 + 12j, 3645 + 2703j), (12 + 3j, 40 + 12j, 96 + 27j, 46852 + 32508j), (40 + 12j, 96 + 27j, 260 + 75j, 1250832 + 892848j), \ldots\ldots, \ldots\}$

2	1	$\{(12+3j,16+3j,56+12j,9081+5271j),(16+3j,56+12j,132+27j,109892+60252j),(56+12j,132+27j,360+75j,3011056+1693488j),\ldots\ldots,\ldots.\}$
3	1	$\{(16+3j,20+3j,72+12j,8703+18517j),(20+3j,72+12j,168+27j,215684+96636j),(72+12j,168+27j,460+75j,6011600+2753328j),\ldots\ldots,\ldots.\}$

B. Sequence Construction of Hyperbolic Diophantine Quadruple Involving Gnomonic number:

Having established the initial Hyperbolic Diophantine pair $(2k+4+3yj,2k+6+3yj)$, this section is to form sequence of triples and extending it to quadruples that preserves the property $D(1)$ across all pairs.

Following the aforementioned algebraic manipulations yield the third tuple t,

$$t = 8k + 20 + 12yj$$

forming Hyperbolic triple (r,s,t). The fourth tuple u follows from conjecture

$$u = 432y^3j^3 + 864ky^2j^2 + 2160y^2j^2 + 576k^2\,yj + 2880kyj + 3588yj + 128k^3 + 960k^2 + 2392k + 1980$$

(r,s,t,u) is a $D(1)$ Hyperbolic Diophantine Quadruple.

For sequence generation the pair (s,t) takes over as basis. The identical algebraic procedure generates the next third tuple v and fourth tuple w, forming Hyperbolic Diophantine quadruple (s,t,v,w) with $D(1)$,

$$v = 18k + 48 + 27yj$$

$$w = 3888y^3j^3 + 7776ky^2j^2 + 2168y^2j^2 + 5184k^2yj + 28224kyj + 38346yj + 1152k^3 + 9408k^2 + 25592k + 23188$$

Subsequently (t,v) becomes the new basis, generating

$$x = 50k + 130 + 75yj$$

$$z = 97200y^3j^3 + 194400ky^2j^2 + 503280y^2j^2 + 129600k^2yj + 671040kyj + 868548yj + 28800k^3 + 223680k^2 + 579032k + 499596$$

Thus, (t,v,x,z) is a $D(1)$ Hyperbolic Diophantine quadruple and this recursive procedure repeat endlessly.

This systematic recursion generates an infinite sequence of $D(1)$ Hyperbolic Diophantine quadruple involving Gnomonic number $\{(r,s,t,u),(s,t,v,w),(t,v,x,z),\ldots\ldots,\ldots.\}$

Few sequences are tabulated below for $y = 1$ and varying k

k	y	$(\{r,s,t,u\}\{s,t,v,w\}\{t,v,x,z\}\ldots\ldots\ldots,\ldots)$.
1	1	$\{(6+3j,8+3j,28+12j,8484+7476j),(8+3j,28+12j,66+27j,88284+75642j),(28+12j,66+27j,2028788+1766388j),\ldots\ldots,\ldots.\}$
2	1	$\{(8+3j,10+3j,36+12j,15516+12084j),(10+3j,36+12j,84+27j,120308+119418j),(36+12j,84+27j,230+75j,3674860+2826228j),\ldots\ldots,\ldots\}$
3	1	$\{(10+3j,12+3j,44+12j,26004+17844j),(12+3j,44+12j,102+27j,260236+173562j),(44+12j,102+27j,280+75j,6113892+4145268j),\ldots\ldots,\ldots..\}$

IV. CONCLUSIONS

This paper establishes a systematic framework for generating infinite sequences of Hyperbolic Diophantine quadruples through algebraic manipulations, presenting concrete results. Future research directions include extension to higher m-tuples, and exploration in cryptographic applications.

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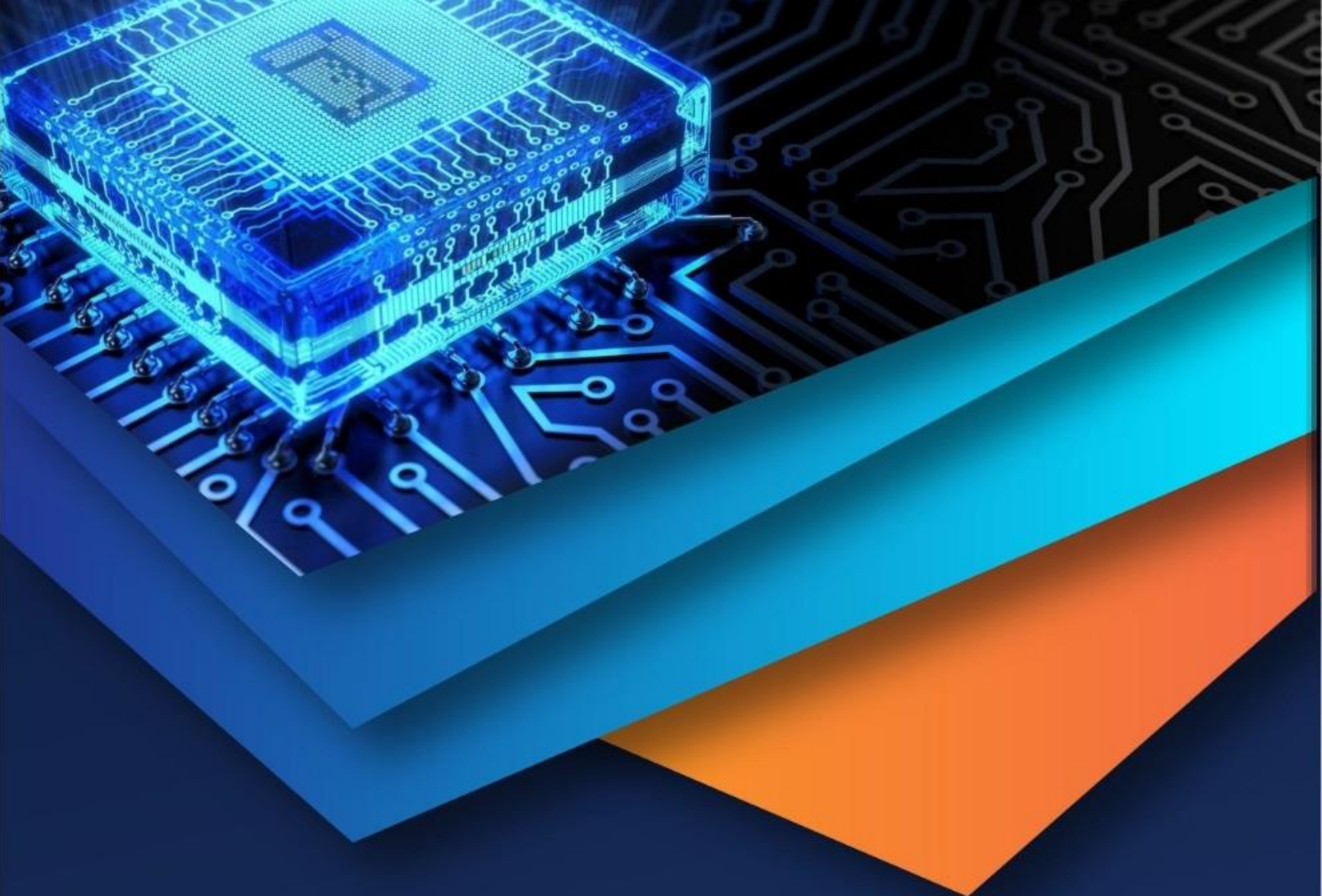

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