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# Enduring Sequences of Hyperbolic Diophantine Quadruples involving Hilbert and Gnomonic Number

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**Abstract:** This paper introduces Hyperbolic Diophantine quadruple, defined as set of four distinct positive integers satisfying the property  $D(n)$ , where pairwise product plus  $n$  yields perfect square. A novel construction generates sequences of such quadruples by first forming a Hyperbolic pair using Hilbert number and Gnomonic number. Algebraic manipulations extend this pair to a triple, then to a quadruple via a conjecture ensuring the fourth element preserves  $D(n)$ .

**Keywords:** Diophantine quadruple, Hyperbolic number, Hilbert number, Gnomonic number, Integer sequence.

**2020 Mathematics Subject classification:** 11D99

## I. INTRODUCTION

Diophantine  $q$ -tuple with the property  $D(n)$  represent a significant extension of the classical problem posed by Diophantus, who sought set of positive rational numbers where the product of any two distinct elements plus a fixed  $n$  yields perfect square. Formally, a set  $(a_1, a_2, \dots, a_q)$  of  $q$  distinct positive integers possess the property  $D(n)$ , for fixed  $n$ , if  $a_i a_j + n$  is a perfect square for all  $1 \leq i < j \leq q$ . While Diophantus identified a quadruple in rational  $(\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16})$  for  $n = 1$ , the integer case has proven more challenging, sparking extensive research into their structure and generation [1-3,9,10]. Constructions of such  $q$ -tuple involving polygonal and pyramidal numbers, which feature prominently in generating triples and quadruples due to their intriguing properties often rely on solutions to Pell equations and Continued fraction expansions [5-8,11-14].

This work pioneers Hyperbolic Diophantine pairs, novel constructs involving Hilbert number and Gnomonic number then extend these pairs to triples preserving the  $D(n)$  condition across all pairs. Conjecture 1 in [4] asserts that every such triple admits a unique fourth tuple, forming a Hyperbolic Diophantine quadruple. Recursive application yields infinite sequence.

## II. PRELIMINARIES

- 1) **Hilbert number:** Hilbert numbers, named after the renowned mathematician David Hilbert, are a specific set of numbers that include all positive integers of the form  $4k + 1, k \geq 0$ .
- 2) **Gnomonic number:** A figurate number of the form  $g_k = 2k - 1$ , giving the area of the square gnomon obtained by removing a square of side  $k - 1$  from a square of side  $k$ .
- 3) **Hyperbolic number:** A Hyperbolic number is an ordered pair of real numbers  $P = p_1 + p_2 j$ , where  $p_1, p_2$  are real numbers and  $j$  is the Hyperbolic unit satisfying  $j^2 = 1$  and  $j \neq \pm 1$ .

## III. SEQUENCE CONSTRUCTION

Consider distinct Hyperbolic numbers denoted as

$$a = \mathcal{H}_k + 3 + 3yj = 4k + 4 + 3yj, \quad b = \mathcal{H}_{k+1} + 3 + 3yj = 4k + 8 + 3yj$$

Their product plus four yields a perfect square that is,

$$ab + 4 = \alpha^2 \tag{3.1}$$

This establishes a Diophantine pair  $(4k + 4 + 3yj, 4k + 8 + 3yj)$  with the property  $D(n)$  for  $n = 4$ .

Similarly, consider

$$r = g_{k+1} + 3 + 3yj = 2k + 4 + 3yj, \quad s = g_{k+2} + 3 + 3yj = 2k + 6 + 3yj$$

such that  $(2k + 4 + 3yj, 2k + 6 + 3yj)$  forms a Diophantine pair for  $n = 1$  satisfying,

$$rs + 1 = \mu^2 \tag{3.2}$$

Algebraic techniques detailed in subsequent subsections systematically extend these pairs through triples to quadruples, thereby generating infinite sequences while preserving the  $D(n)$  throughout the construction process.

*A. Sequence Construction of Hyperbolic Diophantine Quadruple Involving Hilbert number:*

Having established the initial Hyperbolic Diophantine pair  $(4k + 4 + 3yj, 4k + 8 + 3yj)$ , this section is to form sequence of triples and extending it to quadruples that preserves the property  $D(4)$  across all pairs.

For the pair  $(4k + 4 + 3yj, 4k + 8 + 3yj)$ , let  $c$  serve as a third tuple (positive integer)

$$ac + 4 = \beta^2 \tag{3.1.1}$$

$$bc + 4 = \gamma^2 \tag{3.1.2}$$

Assume  $\beta = a + \alpha$  and  $\gamma = b + \alpha$ . Substituting these linear conversions in (3.1.1) and (3.1.2) and deducting leads to

$$(a - b)c = (a + \alpha)^2 - (b + \alpha)^2 \tag{3.1.3}$$

Upon commutating, the value of  $c$  is identified as,

$$c = 16k + 24 + 12yj \tag{3.1.4}$$

$(a, b, c)$  is a Hyperbolic Diophantine triple with  $D(4)$ .

The fourth tuple is subsequently obtained by applying the conjecture 1 [4].

$$d = 81y^3j^3 + 324ky^2j^2 + 477y^2j^2 + 432k^2yj + 1272kyj + 918yj + 192k^3 + 848k^2 + 1224k + 580$$

Thus,  $(a, b, c, d)$  is a Hyperbolic Diophantine quadruple with  $D(4)$ .

To construct sequence, the pair  $(b, c)$  now serves as the basis for progression.

For the pair  $(b, c)$ , let  $e$  be the third tuple such that,

$$be + 4 = \beta^2 \tag{3.1.5}$$

$$ce + 4 = \gamma^2 \tag{3.1.6}$$

Applying linear transformation  $\beta = b + \alpha$ ;  $\gamma = c + \alpha$  in (3.1.5), (3.1.6) and performing the algebraic algorithms yields

$$e = 36k + 60 + 27yj \tag{3.1.7}$$

Forming a Hyperbolic Diophantine triple  $(b, c, e)$

The fourth tuple  $f$  is then obtained as

$$f = 810y^3j^3 + 3240ky^2j^2 + 5544y^2j^2 + 4320k^2yj + 14784kyj + 12594yj + 1920k^3 + 9856k^2 + 16792k + 9500$$

Completing the quadruple  $(b, c, e, f)$  with  $D(4)$ .

The pair  $(c, e)$  now becomes the new basis. The third tuple  $l$  satisfies

$$cl + 4 = \beta^2 \tag{3.1.8}$$

$$el + 4 = \gamma^2 \tag{3.1.9}$$

The corresponding third and fourth tuples are determined as,

$$l = 100k + 160 + 75yj$$

$$m = 24300y^3j^3 + 97200ky^2j^2 + 154440k^2j^2 + 129600k^2yj + 411840kyj + 327108yj + 57600k^3 + 274560k^2 + 436144k + 230888$$

$(c, e, l, m)$  is a  $D(4)$  Hyperbolic Diophantine quadruple.

This recursive process continues indefinitely.

Through this recurring procedure, the sequence of Hyperbolic Diophantine quadruple with  $D(4)$  involving Hilbert number emerges

$\{(a, b, c, d), (b, c, e, f), (c, e, l, m), \dots, \dots\}$

Few sequences are tabulated below for  $y = 1$  and varying  $k$

$k$	$y$	$\{(a, b, c, d)\{b, c, e, f\}\{c, e, l, m\} \dots \dots \dots\}$
1	1	$\{(8 + 3j, 12 + 3j, 40 + 12j, 3645 + 2703j), (12 + 3j, 40 + 12j, 96 + 27j, 46852 + 32508j), (40 + 12j, 96 + 27j, 260 + 75j, 1250832 + 892848j), \dots \dots, \dots\}$

2	1	$\{(12 + 3j, 16 + 3j, 56 + 12j, 9081 + 5271j), (16 + 3j, 56 + 12j, 132 + 27j, 109892 + 60252j), (56 + 12j, 132 + 27j, 360 + 75j, 3011056 + 1693488j), \dots, \dots\}$
3	1	$\{(16 + 3j, 20 + 3j, 72 + 12j, 8703 + 18517j), (20 + 3j, 72 + 12j, 168 + 27j, 215684 + 96636j), (72 + 12j, 168 + 27j, 460 + 75j, 6011600 + 2753328j), \dots, \dots, \dots\}$

**B. Sequence Construction of Hyperbolic Diophantine Quadruple Involving Gnomonic number:**

Having established the initial Hyperbolic Diophantine pair  $(2k + 4 + 3yj, 2k + 6 + 3yj)$ , this section is to form sequence of triples and extending it to quadruples that preserves the property  $D(1)$  across all pairs.

Following the aforementioned algebraic manipulations yield the third tuple  $t$ ,

$$t = 8k + 20 + 12yj$$

forming Hyperbolic triple  $(r, s, t)$ . The fourth tuple  $u$  follows from conjecture

$$u = 432y^3j^3 + 864ky^2j^2 + 2160y^2j^2 + 576k^2yj + 2880kyj + 3588yj + 128k^3 + 960k^2 + 2392k + 1980$$

$(r, s, t, u)$  is a  $D(1)$  Hyperbolic Diophantine Quadruple.

For sequence generation the pair  $(s, t)$  takes over as basis. The identical algebraic procedure generates the next third tuple  $v$  and fourth tuple  $w$ , forming Hyperbolic Diophantine quadruple  $(s, t, v, w)$  with  $D(1)$ ,

$$v = 18k + 48 + 27yj$$

$$w = 3888y^3j^3 + 7776ky^2j^2 + 2168y^2j^2 + 5184k^2yj + 28224kyj + 38346yj + 1152k^3 + 9408k^2 + 25592k + 23188$$

Subsequently  $(t, v)$  becomes the new basis, generating

$$x = 50k + 130 + 75yj$$

$$z = 97200y^3j^3 + 194400ky^2j^2 + 503280y^2j^2 + 129600k^2yj + 671040kyj + 868548yj + 28800k^3 + 223680k^2 + 579032k + 499596$$

Thus,  $(t, v, x, z)$  is a  $D(1)$  Hyperbolic Diophantine quadruple and this recursive procedure repeat endlessly.

This systematic recursion generates an infinite sequence of  $D(1)$  Hyperbolic Diophantine quadruple involving Gnomonic number  $\{(r, s, t, u), (s, t, v, w), (t, v, x, z), \dots, \dots, \dots\}$

Few sequences are tabulated below for  $y = 1$  and varying  $k$

$k$	$y$	$\{(r, s, t, u)\{s, t, v, w\}\{t, v, x, z\} \dots, \dots, \dots\}$
1	1	$\{(6 + 3j, 8 + 3j, 28 + 12j, 8484 + 7476j), (8 + 3j, 28 + 12j, 66 + 27j, 88284 + 75642j), (28 + 12j, 66 + 27j, 2028788 + 1766388j), \dots, \dots, \dots\}$
2	1	$\{(8 + 3j, 10 + 3j, 36 + 12j, 15516 + 12084j), (10 + 3j, 36 + 12j, 84 + 27j, 120308 + 119418j), (36 + 12j, 84 + 27j, 230 + 75j, 3674860 + 2826228j), \dots, \dots, \dots\}$
3	1	$\{(10 + 3j, 12 + 3j, 44 + 12j, 26004 + 17844j), (12 + 3j, 44 + 12j, 102 + 27j, 260236 + 173562j), (44 + 12j, 102 + 27j, 280 + 75j, 6113892 + 4145268j), \dots, \dots, \dots\}$

**IV. CONCLUSIONS**

This paper establishes a systematic framework for generating infinite sequences of Hyperbolic Diophantine quadruples through algebraic manipulations, presenting concrete results. Future research directions include extension to higher m-tuples, and exploration in cryptographic applications.

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