# Eulers Totient Function (Number Theoretic Functions), Right Angle Triangle and Their Applications 

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Abstract: In this paper, we introduce some new theorem and results(section II,III and VI) on Euler's Totient Function, Right angle triangle and their applications
Keywords: Euler's totient function, right angle triangle, Golden ratio, Hardy - Ramanujan number, student's pencil compass and etc.

## I. INTRODUCTION(PRELIMINARY)

1) Definition 1.1 : for $m \geq 1, \varphi(m)$ denote the number of positive integers not exceeding $m$ that are relatively prime to $m$. Euler's totient function is also known as Euler's phi function.
If $m=p_{1} \cdot p_{2}$ where $p_{1}, p_{2}$ are the primes $\left(p_{2}>p_{1}\right)$, Let divisors of $m$ are $1, p_{1}, p_{2}$ and $p_{1} \cdot p_{2}$.
Then,$\varphi(m)=$ (fourth divisors of $m-$ second divisors of $m$ ) - (Third divisor of $m-$ first divisor of $m$ ).

$$
\begin{aligned}
& \text { Or, } \varphi(m)=\left(p_{1} \cdot p_{2}-p_{1}\right)-\left(p_{2}-1\right) \\
& =p_{1}\left(p_{2}-1\right)-\left(p_{2}-1\right) \\
& =\left(p_{1}-1\right)\left(p_{2}-1\right) \\
& \varphi(m)=\left(p_{1}-1\right)\left(p_{2}-1\right) \\
& \text { If } E\left[\operatorname{HCF}\left(1, p_{1}\right), \operatorname{LCM}\left(1, p_{1}\right)\right] \text {, } \\
& F\left[\operatorname{HCF}\left(p_{1}, p_{2}\right), \operatorname{LCM}\left(p_{1}, p_{2}\right)\right] \text {, and } \\
& G\left[\operatorname{HCF}\left(p_{2}, p_{1} \cdot p_{2}\right), \operatorname{LCM}\left(p_{2}, p_{1} \cdot p_{2}\right)\right] \\
& \text { Are the vertices of triangle EFG }
\end{aligned}
$$


F G
E


Then triangle EFG is inverted right angle triangle

$$
\text { And } \varphi(m)=|A-B|
$$

Where $\mathrm{A}, \mathrm{B}$ are the Altitude and Base of triangle EFG

$$
\begin{aligned}
& \text { and (i) If } A>B \\
& \text { then, } A=p_{1} \cdot p_{2}-p_{1} \text { and } B=p_{2}-1 \\
& \text { (ii) If } B>A \\
& \text { then, } B=p_{1} \cdot p_{2}-p_{1} \text { and } A=p_{2}-1 \\
& \varphi(m)=|A-B|
\end{aligned}
$$

Squaring the both side,

$$
(\varphi(m))^{2}=A^{2}+B^{2}-2 \cdot A \cdot B
$$

Since, $A^{2}+B^{2}=H^{2}$ where H is the hypotenuse

$$
\text { Then, }(\varphi(m))^{2}=H^{2}-2 \cdot A \cdot B
$$

divide both side of above equation by 4 ,

$$
\begin{aligned}
& \left(\frac{\varphi(m)}{2}\right)^{2}=\left(\frac{H}{2}\right)^{2}-\frac{1}{2} \cdot \mathrm{~A} \cdot \mathrm{~B} \\
& \text { Or, } \mathrm{T}=\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(m)}{2}\right)^{2}
\end{aligned}
$$

Where T is the area of triangle $\mathrm{EFG}=\frac{1}{2} \cdot A \cdot B$.
$H^{n}-(\varphi(m))^{n}$ is perfectly divisible by $2^{n}, n \geq 1$.
If $m=p_{1} \cdot p_{2} \ldots p_{z}, Z \geq 2$ then total pair of primes $=\frac{z(z-1)}{2} \operatorname{Or}(Z-1)^{\text {th }}$ triangular number
Therefore, total number of inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$ is $(\mathrm{z}-1)^{\text {th }}$ Triangular number.
Theorem and Results

## II. INEQUALITY RELATION

1) Theorem 2.1: If $=p_{1} \cdot p_{2} \ldots p_{z}$,

Where $p_{1}, p_{2}, \ldots . p_{z}$ are the primes

$$
\left(p_{1}<p_{2}<\ldots .<p_{z}\right), Z \geq 3 .
$$

Let divisors of m are $\mathrm{d}_{1}, \mathrm{~d}_{2}, \ldots . . \mathrm{d}_{\tau(\mathrm{m})}$, Consider m have ${ }^{\tau(\mathrm{m})} \mathrm{c}_{4}\left[{ }^{\tau(\mathrm{m})} \mathrm{c}_{4}=\sum_{i=i}^{n} i\left[\{\tau(\mathrm{~m})-2-\mathrm{i}\}^{\text {th }}\right.\right.$ triangular number $]$, where $\left.\mathrm{n}=\tau(\mathrm{m})-3\right]$ groups, each group consists of four elements ( at least one element of each group should be distinct ) . let groups be (a, b, c, d'), $\left(a^{\prime \prime}, b^{\prime \prime}, c^{\prime \prime}, d^{\prime \prime}\right), \ldots . .\left(a^{\prime \prime . . e t i m e s}, b^{\prime \prime} . . e\right.$ times,$c^{\prime \prime} \ldots$ e times,$d^{\prime \prime} \ldots$ etimes $)$ where $e={ }^{\tau(m)} c_{4}$

And every four elements of each group belongs to the divisors of $m$ that is $d_{1}, d_{2}, \ldots \ldots d_{\tau(m)}$. Let $\mathrm{E}_{1}\left[\mathrm{HCF}\left(\mathrm{a}^{\prime}, \mathrm{b}^{\prime}\right), \operatorname{LCM}\left(\mathrm{a}^{\prime}, \mathrm{b}^{\prime}\right)\right]$,
$\mathrm{F}_{1}\left[\operatorname{HCF}\left(\mathrm{~b}^{\prime}, \mathrm{c}^{\prime}\right), \operatorname{LCM}\left(\mathrm{b}^{\prime}, \mathrm{c}^{\prime}\right)\right], \mathrm{G}_{1}\left[\operatorname{HCF}\left(\mathrm{c}^{\prime}, \mathrm{d}^{\prime}\right), \operatorname{LCM}\left(\mathrm{c}^{\prime}, \mathrm{d}^{\prime}\right)\right], \mathrm{E}_{2}\left[\operatorname{HCF}\left(\mathrm{a}^{\prime \prime}, \mathrm{b}^{\prime}\right), \operatorname{LCM}\left(\mathrm{a}^{\prime \prime}, \mathrm{b}^{\prime}\right)\right]$,

 points) of $\mathrm{E}_{1} \mathrm{~F}_{1} \mathrm{G}_{1}, \mathrm{E}_{2} \mathrm{~F}_{2} \mathrm{G}_{2}, \ldots . \mathrm{E}_{\mathrm{e}} \mathrm{F}_{\mathrm{e}} \mathrm{G}_{\mathrm{e}}$ and
$\mathrm{W}_{1}=$ Total number of groups for which $\mathrm{E}_{\Psi}, \mathrm{F}_{\Psi}, \mathrm{G}_{\Psi}$ form Collinear line.
Where $\Psi$ belongs to [1,e]
$\mathrm{W}_{2}=$ Total number of groups for which $\mathrm{E}_{\Psi}, \mathrm{F}_{\Psi}, \mathrm{G}_{\Psi}$ form triangles (other than right angle triangle).
$\mathrm{W}_{3}=$ Total number of groups for which $\mathrm{E}_{\Psi}, \mathrm{F}_{\Psi}, \mathrm{G}_{\Psi}$ form non-inverted right angle triangles


$\mathrm{W}_{4}=$ Total number of groups for which $\mathrm{E}_{\Psi}, \mathrm{F}_{\Psi}, \mathrm{G}_{\Psi}$ form inverted right angle triangles whose area is not equal to $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$ and $\mathrm{W}_{5}=$ Total number of groups for which $\mathrm{E}_{\Psi}, \mathrm{F}_{\Psi}, \mathrm{G}_{\Psi}$ form inverted right angle triangles whose area is equal to $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$ where $\mathrm{S} \neq \mathrm{m}$.

 ${ }^{1} \mathrm{C}_{2}$ inverted right angle triangles whose area is $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$.
That is $\mathrm{w}_{5} \geq \mathrm{Z}-1+{ }^{\mathrm{Z}-1} \mathrm{C}_{2}$ or, $\sum_{i=1}^{n} i\left[\{\tau(\mathrm{~m})-2-\mathrm{i}\}^{\text {th }}\right.$ triangular number $]-\sum_{j=1}^{4} w j \geq \mathrm{Z}-1+{ }^{\mathrm{Z}-\mathrm{l}} \mathrm{C}_{2}$, where $\mathrm{n}=\tau(\mathrm{m})-3$.
Where total number of groups $=\mathrm{W}_{1}+\mathrm{W}_{2}+\mathrm{W}_{3}+\mathrm{W}_{4}+\mathrm{W}_{5}$.
Proof: If $m=P_{1} \cdot P_{2} \ldots . \mathrm{P}_{\mathrm{Z}}, \mathrm{Z} \geq 3$.
Total pair of primes $=\mathrm{Z}-1+{ }^{\mathrm{Z}-1} \mathrm{C}_{2}$ and number of groups of the form $\left(1, \mathrm{P}^{\prime}, \mathrm{P}^{\prime}, \mathrm{P}^{\prime}, \mathrm{P}^{\prime \prime}\right)$ is $\mathrm{Z}-1+{ }^{\mathrm{Z}-1} \mathrm{C}_{2}$
Where $P^{\prime}$ and $P^{\prime \prime}$ are one of $P_{1}, P_{2}, \ldots . P_{Z}\left(P^{\prime}<P^{\prime}{ }^{\prime}\right)$. If groups are of the form ( $1, \mathrm{P}^{\prime}, \mathrm{P}^{\prime}, \mathrm{P}^{\prime}, \mathrm{P}^{\prime \prime}$ ) then $\mathrm{E}, \mathrm{F}, \mathrm{G}$ form inverted right angle triangles, and every $\mathrm{Z}-1+{ }^{\mathrm{Z}-1} \mathrm{C}_{2}$ triangles follow the relation $\mathrm{T}=\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$. Where $\mathrm{S}=\mathrm{p} . \mathrm{p}$ ". By considering the groups of four elements (elements belongs to the divisors of $m$ ) , there exist some other groups also which are not of the form ( $1, \mathrm{P}^{\prime}, \mathrm{P}^{\prime}, \mathrm{P}^{\prime} . \mathrm{P}^{\prime \prime}$ ) but follow the relation $\mathrm{T}=\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$. If $\mathrm{m}=\mathrm{P}_{1} \cdot \mathrm{P}_{2} \cdot \mathrm{P}_{3}$, then total number of groups $={ }^{\tau(\mathrm{m})} \mathrm{c}_{4}={ }^{8} \mathrm{C}_{4}=70$.

Out of these 70 groups there exist definitely $\mathrm{Z}-1+{ }^{\mathrm{Z}-1} \mathrm{C}_{2}$, that is 3 groups for which the vertices form inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$.
Above 3 groups are of the form ( $\left.1, \mathrm{P}^{\prime}, \mathrm{P}^{\prime}{ }^{\prime}, \mathrm{P}^{\prime} \cdot \mathrm{P}^{\prime \prime}\right)$ where $\mathrm{P}^{\prime}$ and $\mathrm{P}^{\prime \prime}$ are one of $\mathrm{P}_{1}, \mathrm{P}_{2}, \mathrm{P}_{3}\left(\mathrm{P}^{\prime}<\mathrm{P}^{\prime \prime}\right)$. But for some m there exist some other groups which are not of the form ( $1, \mathrm{P}^{\prime}, \mathrm{P}^{\prime}, \mathrm{P}^{\prime} . \mathrm{P}^{\prime \prime}$ ) but follow the relation $\mathrm{T}=\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$.
If $m=2 . P_{2} \cdot P_{3}$, then the vertices ( $\mathrm{E}, \mathrm{F}, \mathrm{G}$ ) for the group (2,2.p $2,2 \cdot \mathrm{p}_{3}, 2 . \mathrm{p}_{2} \cdot \mathrm{p}_{3}$ ) form inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^{2}-$ $\left(\frac{\varphi(S)}{2}\right)^{2}$.
So, if $\mathrm{m}=2 . \mathrm{P}_{2} \cdot \mathrm{P}_{3}$, then there exist 4 inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$.
If $\mathrm{m}=3 . \mathrm{P}_{2} \cdot \mathrm{P}_{3}$, then the vertices ( $\mathrm{E}, \mathrm{F}, \mathrm{G}$ ) for the group $\left(3,3 \cdot \mathrm{P}_{2}, 3 \cdot \mathrm{P}_{3}, 3 \cdot \mathrm{P}_{2} \cdot \mathrm{P}_{3}\right)$ form inverted right angle triangle but area is not equal to $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$.
So, If $\mathrm{m}=3 . \mathrm{P}_{2} \cdot \mathrm{P}_{3}$ then there exist 3 inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$.
Hence, there exist at least $\mathrm{Z}-1+{ }^{\mathrm{Z}-\mathrm{l}} \mathrm{C}_{2}$ inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$.

(b) let $\mathrm{m}=2.3 .5$ then , groups are $(1,2,3,5),(1,2,3,6),(1,2,3,10),(1,2,3,15),(1,2,3,30),(1,2,5,6),(1,2,5,10),(1,2,5,15)$, $(1,2,5,30),(1,2,6,10),(1,2,6,15),(1,2,6,30),(1,2,10,15),(1,2,10,30),(1,2,15,30),(1,3,5,6),(1,3,5,10),(1,3,5,15)$,
$\ldots \ldots \ldots \ldots \ldots \ldots . .,(6,10,15,30)$. If $m=2.3 .5$, then total number of groups $=70$.
(c) $d_{1}=a=1$.

Theorem 2.1 in another form :
A lot consists of groups for which $\mathrm{E}_{\Psi}, \mathrm{F}_{\Psi}, \mathrm{G}_{\Psi}$ form inverted right angle triangle, then probability ( $q_{1}$ ) of getting the inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$
is greater than or equal to $\frac{z-1+^{z-1} c_{2}}{w_{4}+w_{5}}$

Or, $q_{1} \geq \frac{z-1+^{z-1} c_{2}}{w_{4}+w_{5}}$ and probability $\left(q_{2}\right)$ of getting the inverted right angle triangle whose area is not equal to
$\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$ is less than or equal to $\frac{w_{4}+w_{5}{ }^{-1-1} c_{2-z+1}}{w_{4}+w_{5}}$
Or, $q_{2} \leq \frac{w_{4}+w_{5}{ }^{-1-1} c_{2-z+1}}{w_{4}+w_{5}}$.
$q_{1}$ can also be written as ,

$$
q_{1} \geq \frac{z-1+{ }^{z-1} c_{2}}{\sum_{i=i}^{n} i[\{\tau(\mathrm{~m})-2-\mathrm{i}\} \text { th triangular number }]-\sum_{j=1}^{3} w_{j}}
$$

Note: If $m=p_{1} \cdot p_{2} \cdot p_{3}$ then,
$q_{1} \geq 0.1364$ and $q_{2} \leq 0.8637$.

## III. RESULTS RELATED TO HARDY-RAMANUJAN NUMBER, AREA OF RECTANGLE AND TRAPEZIUM AND GOLDEN RATIO .

1) Result 3.1: If $m=3.5 .7$, out of the 70 groups if we take $27^{\text {th }}$ and $41^{\text {th }}$ groups that are $(1,7,15,35)$ and $(3,5,15,35)$ and if $E_{27}[\operatorname{HCF}(1,7), \operatorname{LCM}(1,7)], F_{27}[\operatorname{HCF}(7,15), \operatorname{LCM}(7,15)], G_{27}[\operatorname{HCF}(15,35), \operatorname{LCM}(15,35)]$
and $E_{41}[H C F(3,5), \operatorname{LCM}(3,5)]$,
$F_{41}[\operatorname{HCF}(5,15), \operatorname{LCM}(5,15)]$,
$G_{41}[\operatorname{HCF}(15,35), L C M(15,35)]$ are the vertices of triangles $\mathrm{E}_{27}, \mathrm{~F}_{27}, \mathrm{G}_{27}$ and $\mathrm{E}_{41} \mathrm{~F}_{41} \mathrm{G}_{41}$, then the average of $\left(\frac{H_{27}}{2}\right)^{2}-\left(\frac{\varphi\left(S_{27}\right)}{2}\right)^{2}$ And $\left(\frac{H_{41}}{2}\right)^{2}-\left(\frac{\varphi\left(S_{41}\right)}{2}\right)^{2}$ is Hardy - Ramanujan number
Where $\mathrm{E}_{27} \mathrm{~F}_{27} \mathrm{G}_{27}$ is inverted right angle triangle, $\mathrm{E}_{41} \mathrm{~F}_{41} \mathrm{G}_{41}$ is non-inverted right angle triangle. $\mathrm{H}_{27}, \mathrm{H}_{41}$ are the length of hypotenuse of triangles $\mathrm{E}_{27} \mathrm{~F}_{27} \mathrm{G}_{27}$ and $\mathrm{E}_{41} \mathrm{~F}_{41} \mathrm{G}_{41}$
$\mathrm{S}_{27}=7 \times 15, \mathrm{~S}_{41}=5 \times 15$ and area of both triangle is not equal to $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$.
$\frac{\left(\frac{H_{27}}{2}\right)^{2}-\left(\frac{\varphi\left(S_{27}\right.}{2}\right)^{2}+\left(\frac{H_{41}}{2}\right)^{2}-\left(\frac{\varphi\left(S_{41}\right.}{2}\right)^{2}}{2}=1729$.
2) Result 3.2: If $\mathrm{m}=\mathrm{P}_{1} \cdot \mathrm{P}_{2}\left(\mathrm{P}_{1}<\mathrm{P}_{2}\right)$, divisors of m are $1, \mathrm{P}_{1}, \mathrm{P}_{2}, \mathrm{P}_{1} . \mathrm{P}_{2}$.

Let $\mathrm{E}\left[\operatorname{HCF}\left(1, \mathrm{P}_{1}\right), \operatorname{LCM}\left(1, \mathrm{P}_{1}\right)\right]$,
$\mathrm{F}\left[\operatorname{HCF}\left(\mathrm{P}_{1}, \mathrm{P}_{2}\right), \operatorname{LCM}\left(\mathrm{P}_{1}, \mathrm{P}_{2}\right)\right]$, and
$\mathrm{G}\left[\operatorname{HCF}\left(\mathrm{P}_{2}, \mathrm{P}_{1} \cdot \mathrm{P}_{2}\right), \operatorname{LCM}\left(\mathrm{P}_{2}, \mathrm{P}_{1} \cdot \mathrm{P}_{2}\right)\right]$ are the vertices of triangle EFG


Let Area of Rectangle $S^{\prime} U^{\prime} E F=R$ and Area of trapezium
$V^{\prime} W^{\prime}$ GE $=\mathrm{K}$ then,
(a) $\quad \varphi(\mathrm{m})=\mathrm{R}-\sqrt{H^{2}-R^{2}}$
(b) $\quad \varphi(\mathrm{m})=2 \mathrm{R}-\sqrt{H^{2}+4(K-R)}$
where H is the length of hypotenuse of right angle triangle EFG .
Proof: $\mathrm{EG}=\sqrt{\left(p_{2}-1\right)^{2}+\left(p_{1} \cdot p_{2}-p_{1}\right)^{2}}$

$$
\begin{aligned}
& =\left(p_{2}-1\right) \cdot \sqrt{1+p_{1}{ }^{2}} \\
\mathrm{EG} & =\mathrm{H}=\left(p_{2}-1\right) \cdot \sqrt{1+p_{1}{ }^{2}}
\end{aligned}
$$

Area of Rectangle $=\mathrm{S}^{\prime} \mathrm{U}^{\prime} \mathrm{EF}=\mathrm{R}=p_{1}\left(p_{2}-1\right)$
Area of Trapezium $=V^{\prime} W^{\prime}$ GE $=$
$\mathrm{K}=\frac{1}{2} p_{1}\left(p_{2}-1\right)\left(p_{2}+1\right)$
Put $\mathrm{R}=p_{1}\left(p_{2}-1\right)$ and $\mathrm{H}=\left(p_{2}-1\right) \cdot \sqrt{1+{p_{1}^{2}}^{2}}$ in (a) that is $\mathrm{R}-\sqrt{H^{2}-R^{2}}$, We get

$$
\begin{aligned}
& \quad p_{1}\left(p_{2}-1\right)-\sqrt{\left(p_{2}-1\right)^{2}\left(1+p_{1}^{2}\right)-p_{1}^{2}\left(p_{2}-1\right)^{2}} \\
& =p_{1}\left(p_{2}-1\right)-\left(p_{2}-1\right) \\
& =\left(p_{2}-1\right)\left(p_{1}-1\right) \\
& = \\
& =\varphi(m)
\end{aligned}
$$

Hence (a) part is proved.
Now, put $\mathrm{R}=p_{1}\left(p_{2}-1\right), \mathrm{H}=\left(p_{2}-1\right) \sqrt{1+p_{1}{ }^{2}}$
And $\mathrm{K}=\frac{1}{2} p_{1}\left(p_{2}-1\right)\left(p_{2}+1\right)$ in (b) that is
$2 R-\sqrt{H^{2}+4(K-R)}$, we get
$=2 \mathrm{p}_{1}\left(\mathrm{p}_{2}-1\right)-\left[\left(\mathrm{p}_{2}-1\right)^{2}\left(1+\mathrm{p}_{1}{ }^{2}\right)+2\left\{\mathrm{p}_{1}\left(\mathrm{p}_{2}-1\right)\left(\mathrm{p}_{2}+1\right)-2 \mathrm{p}_{1}\left(\mathrm{p}_{2}-1\right)\right\}\right]^{1 / 2}$

$$
\begin{aligned}
& =2 p_{1}\left(p_{2}-1\right)-\left[\left(p_{2}-1\right)^{2}\left(1+p_{1}^{2}\right)+2 p_{1}\left(p_{2}-1\right)\left(p_{2}+1-2\right)\right]^{1 / 2} \\
& =2 p_{1}\left(p_{2}-1\right)-\left[\left(p_{2}-1\right)^{2}\left(1+p_{1}^{2}\right)+2 p_{1}\left(p_{2}-1\right)^{2}\right]^{1 / 2} \\
& =2 p_{1}\left(p_{2}-1\right)-\left[\left(p_{2}-1\right)^{2}\left(1+p_{1}^{2}+2 p_{1}\right)\right]^{1 / 2} \\
& =2 p_{1}\left(p_{2}-1\right)-\left(p_{2}-1\right)\left(p_{1}+1\right) \\
& =\left(p_{2}-1\right)\left[2 p_{1}-p_{1}-1\right] \\
& =\left(p_{2}-1\right)\left(p_{1}-1\right) \\
& =\left(p_{1}-1\right)\left(p_{2}-1\right) \\
& =\varphi(m)
\end{aligned}
$$

Hence (b) part is proved.
3) Result 3.3: If $\mathrm{m}=\mathrm{p}_{1} \cdot \mathrm{p}_{2}$ where $\mathrm{p}_{1}, \mathrm{p}_{2}$ are the primes $\left(\mathrm{p}_{1}<\mathrm{p}_{2}\right)$, divisors of m are $1, \mathrm{p}_{1}, \mathrm{p}_{2}, \mathrm{p}_{1} \cdot \mathrm{p}_{2}$. let $\mathrm{E}\left[\operatorname{HCF}\left(1, \mathrm{p}_{1}\right), \operatorname{LCM}\left(1, \mathrm{p}_{1}\right)\right], \mathrm{F}$ $\left[\operatorname{HCF}\left(\mathrm{p}_{1}, \mathrm{p}_{2}\right), \operatorname{LCM}\left(\mathrm{p}_{1}, \mathrm{p}_{2}\right)\right]$ and $\mathrm{G}\left[\operatorname{HCF}\left(\mathrm{p}_{2}, \mathrm{p}_{1} . \mathrm{p}_{2}\right), \operatorname{LCM}\left(\mathrm{p}_{2}, \mathrm{p}_{1} . \mathrm{p}_{2}\right)\right]$ are the vertices of triangle EFG , then roots of the quadratic equation
2. $\left(\varphi\left(p_{1}\right)\right)^{2} x^{2}-2\left(\varphi\left(p_{1}\right)\right)^{2} x^{1}-p_{1}=0 \quad \operatorname{are} \frac{\sqrt{H^{2}-4 T} \pm H}{2 \sqrt{H^{2}-4 T}}$

Proof :- 2. $\left(\varphi\left(p_{1}\right)\right)^{2} x^{2}-2\left(\varphi\left(p_{1}\right)\right)^{2} x^{1}-p_{1}=0$

$$
x=\frac{\left(p_{1}-1\right) \pm \sqrt{1+p_{1}^{2}}}{2\left(p_{1}-1\right)}
$$

Since, $\mathrm{H}=\left(p_{2}-1\right) \sqrt{1+p_{1}{ }^{2}}$ and $\mathrm{T}={ }_{2}^{1} \mathrm{p}_{1}\left(\mathrm{p}_{2}-1\right)^{2}$
Where T is the area of triangle EFG

$$
\frac{\sqrt{H^{2}-4 T} \pm H}{2 \sqrt{H^{2}-4 T}}=\frac{\left(p_{1}-1\right) \pm \sqrt{1+p_{1}^{2}}}{2\left(p_{1}-1\right)}
$$

Hence, Proved.
4) Result 3.4:

$$
\text { If } m=p_{1} \cdot p_{2} \text { where } p_{1} \text { is either } 2 \text { or } 3 \text { and }
$$

$p_{1}<p_{2}$, Divisors of $m$ are $1, p_{1}, p_{2}, p_{1} \cdot p_{2}$.

$$
\text { Let } E\left[\operatorname{HCF}\left(1, p_{1}\right), \operatorname{LCM}\left(1, p_{1}\right)\right], F\left[\operatorname{HCF}\left(p_{1}, p_{2}\right), \operatorname{LCM}\left(p_{1}, p_{2}\right)\right.
$$

And $G\left[\operatorname{HCF}\left(p_{2}, p_{1} \cdot p_{2}\right), \operatorname{LCM}\left(p_{2}, p_{1} \cdot p_{2}\right)\right]$ are the vertices of triangle EFG , then
2. $\lim _{n \rightarrow \infty}\left(\frac{U_{n+1}}{U_{n}}\right)-1=\sqrt{\frac{4 T+(\varphi(m))^{2}}{\varphi\left(p_{2}\right) \cdot \varphi(m)}}=\frac{H}{\varphi\left(p_{2}\right) \cdot \sqrt{\varphi\left(p_{1}\right)}}$
where $T$ is the area of triangle $E F G$.
Proof: If $m=p_{1} \cdot p_{2}$, then $\varphi(m)=\left(p_{1}-1\right) \cdot\left(p_{2}-1\right)$,


Vertices of above triangle EFG are $\mathrm{E}\left(1, \mathrm{p}_{1}\right), \mathrm{F}\left(1, \mathrm{P}_{1} \cdot \mathrm{P}_{2}\right)$ and $\mathrm{G}\left(\mathrm{P}_{2}, \mathrm{P}_{1} \cdot \mathrm{P}_{2}\right)$.

$$
\begin{aligned}
T & =\frac{1}{2} \cdot E F \cdot F G \\
O r, T & =\frac{1}{2} \cdot p_{1} \cdot\left(p_{2}-1\right)^{2}
\end{aligned}
$$

Put $T$ and $\varphi(m)$ in R.H.S,

$$
\sqrt{\frac{2 p_{1}\left(p_{2}-1\right)^{2}+\left(p_{1}-1\right)^{2}\left(p_{2}-1\right)^{2}}{\left(p_{1}-1\right)\left(p_{2}-1\right)^{2}}}=\sqrt{\frac{p_{1}^{2}+1}{p_{1}-1}}
$$

(i)

$$
\text { If } p_{1}=2 \text {, then } \sqrt{\frac{p_{1}^{2}+1}{p_{1}-1}}=\sqrt{5}
$$

(ii)

$$
\text { If } p_{1}=3 \text {, then } \sqrt{\frac{p_{1}^{2}+1}{p_{1}-1}}=\sqrt{5}
$$

$$
\begin{gathered}
\text { Since, for } n \geq 1 \lim _{n \rightarrow \infty}\left(\frac{U_{n+1}}{U_{n}}\right)=\frac{1+\sqrt{5}}{2} \\
\text { Or, } 2 . \lim _{n \rightarrow \infty}\left(\frac{U_{n+1}}{U_{n}}\right)-1=\sqrt{5}
\end{gathered}
$$

Hence, $2 . \lim _{n \rightarrow \infty}\left(\frac{U_{n+1}}{U_{n}}\right)-1=\sqrt{\frac{4 T+(\varphi(m))^{2}}{\varphi\left(p_{2}\right) \cdot \varphi(m)}}=\frac{H}{\varphi\left(p_{2}\right) \cdot \sqrt{\varphi\left(p_{1}\right)}}$.

## IV. ANGLES OF TRIANGLE EFG IN TERMS OF EULERS PHI FUNCTION

1) Theorem 4.1: If $m=p_{1} \cdot \mathrm{p}_{2}$ where $\mathrm{p}_{1}, \mathrm{p}_{2}$ are the primes ( $\mathrm{p}_{1}<\mathrm{p}_{2}$ ), divisors of m are $1, \mathrm{p}_{1}, \mathrm{p}_{2}, \mathrm{p}_{1} \cdot \mathrm{p}_{2}$. let $\mathrm{E}\left[\operatorname{HCF}\left(1, \mathrm{p}_{1}\right), \operatorname{LCM}\left(1, \mathrm{p}_{1}\right)\right], \mathrm{F}$ $\left[\operatorname{HCF}\left(\mathrm{p}_{1}, \mathrm{p}_{2}\right), \operatorname{LCM}\left(\mathrm{p}_{1}, \mathrm{p}_{2}\right)\right]$ and $\operatorname{G}\left[\operatorname{HCF}\left(\mathrm{p}_{2}, \mathrm{p}_{1} \cdot \mathrm{p}_{2}\right), \operatorname{LCM}\left(\mathrm{p}_{2}, \mathrm{p}_{1} \cdot \mathrm{p}_{2}\right)\right]$ are the vertices of triangle $\operatorname{EFG}$ and angle $\mathrm{E}=\alpha$ and angle G $=\beta$, then the angles of triangle EFG (in terms of euler's phi function ) are $\alpha=\frac{1}{2} \cot ^{-1}\left[\frac{\varphi(m) \sqrt{2 H^{2}-(\varphi(m))^{2}}}{H^{2}-(\varphi(m))^{2}}\right]$ and
$\beta=\frac{1}{2} \cot ^{-1}\left[\frac{\varphi(m) \sqrt{2 H^{2}-(\varphi(m))^{2}}}{(\varphi(m))^{2}-H^{2}}\right]$

Proof: Put $\varphi(m)=\left(p_{1}-1\right)\left(p_{2}-1\right)$ and
$H=\left(p_{2}-1\right) \sqrt{1+p_{1}{ }^{2}}$ in
$\alpha=\frac{1}{2} \cot ^{-1}\left[\frac{\varphi(m) \sqrt{2 H^{2}-(\varphi(m))^{2}}}{H^{2}-(\varphi(m))^{2}}\right]$,
$=\frac{1}{2} \cot ^{-1}\left[\frac{\left(p_{1}-1\right)\left(p_{2}-1\right)\left\{2\left(p_{2}-1\right)^{2}\left(1+p_{1}^{2}\right)-\left(p_{1}-1\right)^{2}\left(p_{2}-1\right)^{2}\right\}^{1 / 2}}{\left(p_{2}-1\right)^{2}\left(1+p_{1}^{2}\right)-\left(p_{1}-1\right)^{2}\left(p_{2}-1\right)^{2}}\right]$
$=\frac{1}{2} \cot ^{-1}\left[\frac{\left(p_{1}-1\right)\left\{2\left(1+p_{1}{ }^{2}\right)-\left(p_{1}-1\right)^{2}\right\}^{1 / 2}}{\left\{\left(1+p_{1}{ }^{2}\right)-\left(p_{1}-1\right)^{2}\right\}}\right]$
$=\frac{1}{2} \cot ^{-1}\left[\frac{\left(p_{1}-1\right)\left\{2+2 p_{1}{ }^{2}-p_{1}{ }^{2}-1+2 p_{1}\right\}^{1 / 2}}{1+p_{1}{ }^{2}-p_{1}{ }^{2}-1+2 p_{1}}\right]$
$=\frac{1}{2} \cot ^{-1}\left[\frac{\left(p_{1}-1\right)\left\{p_{1}^{2}+2 p_{1}+1\right\}^{1 / 2}}{2 p_{1}}\right]$
$=\frac{1}{2} \cot ^{-1}\left[\frac{\left(p_{1}-1\right)\left(p_{1}+1\right)}{2 p_{1}}\right]$
$=\frac{1}{2} \cot ^{-1}\left[\frac{p_{1}{ }^{2}-1}{2 p_{1}}\right]=\frac{1}{2} \tan ^{-1}\left[\frac{2 p_{1}}{p_{1}{ }^{2}-1}\right]$
Since, $\tan \alpha=\frac{G F}{E F}$
Or, $\tan \alpha=\frac{p_{2}-1}{p_{1}\left(p_{2}-1\right)}=\frac{1}{p_{1}}$
From (*) equation,

$$
\frac{1}{2} \tan ^{-1}\left[\frac{2 p_{1}}{p_{1}^{2}-1}\right]=\alpha
$$

Or, $\tan (2 \alpha)=\frac{2 p_{1}}{p_{1}^{2}-1}$
Or, $\frac{2 \tan \alpha}{1-\tan ^{2} \alpha}=\frac{2 p_{1}}{p_{1}^{2}-1}$
Put $\alpha=\frac{1}{p_{1}}$,

$$
\frac{2 \times \frac{1}{p_{1}}}{1-\frac{1}{p_{1}^{2}}}=\frac{2 p_{1}}{p_{1}^{2}-1}
$$

L.H.S = R.H.S

Hence, Proved.
Since, $\alpha+\beta=\frac{\pi}{2}$

$$
\begin{aligned}
& \frac{1}{2} \tan ^{-1}\left[\frac{2 p_{1}}{p_{1}^{2}-1}\right]+\beta=\frac{\pi}{2} \\
& \beta=\frac{\pi}{2}-\frac{1}{2} \tan ^{-1}\left[\frac{2 p_{1}}{p_{1}{ }^{2}-1}\right]
\end{aligned}
$$

$$
\beta=\frac{1}{2}\left[\pi-\tan ^{-1}\left\{\frac{2 p_{1}}{p_{1}{ }^{2}-1}\right\}\right]
$$

Or, $\beta=\frac{1}{2}\left[\pi-\cot ^{-1}\left\{\frac{p^{2}-1}{2 p_{1}}\right\}\right]$

$$
\beta=\frac{1}{2}\left[\cot ^{-1}\left\{\frac{1-p_{1}^{2}}{2 p_{1}}\right\}\right]
$$

Hence, $\beta=\frac{1}{2} \cot ^{-1}\left[\frac{\varphi(m) \sqrt{2 H^{2}-(\varphi(m))^{2}}}{(\varphi(m))^{2}-H^{2}}\right]$

## V. OBSERVATIONS

(5.1) If $m=2.3 .5$, then $w_{4}=18, w_{5}=4$, and $w_{4}+w_{5}=22$.

If $\mathrm{m}=3.5 .7$, then $\mathrm{w}_{4}=19, \mathrm{w}_{5}=3$, and $\mathrm{w}_{4}+\mathrm{w}_{5}=22$
So, $w_{4}+w_{5}$ is constant but $w_{4}, w_{5}$ are not constant.
If $\mathrm{m}=\mathrm{p}_{1} \cdot \mathrm{p}_{2} . \mathrm{p}_{3}$, for different $\mathrm{m} w_{1}, w_{2}, w_{3}$ and $w_{4}+w_{5}$ are constants.
$w_{1}=10, w_{2}=36, w_{3}=2$ and $w_{4}+w_{5}=22$.

$$
w_{2}>w_{4}+w_{5}>w_{1}>w_{3} .
$$

Total number of right angle triangles $=w_{3}+w_{4}+w_{5}$
$=\varphi\left(\sum_{k=1}^{5} w_{k}\right)=\varphi\left(8_{C_{4}}\right)=$
$\varphi\left[\sum_{i=i}^{5} i[\{\tau(m)-2-i\} t h\right.$ triangular number $\left.]\right]=24$.
Or, $w_{3}+w_{4}+w_{5}=\varphi\left(w_{1}+w_{2}+w_{3}+w_{4}+w_{5}\right)$
Or, $\sum_{k=1}^{5} w_{k}-\varphi\left(\sum_{k=1}^{5} w_{k}\right)=w_{1}+w_{2}$
That is $70-24=10+36$
From Theorem 2.1: $w_{5} \geq z-1+{ }^{z-1} c_{2}$
If $m=p_{1} \cdot p_{2} \cdot p_{3}$, then,
$\varphi\left(\sum_{k=1}^{5} w_{k}\right) \geq z-1+{ }^{z-1} c_{2}+w_{3}+w_{4}$, Where $\sum_{k=1}^{5} w_{k}=$ Total No. of groups.
And $\varphi\left(\sum_{k=1}^{5} w_{k}\right)-z+1-^{z-1} c_{2}-w_{3} \geq w_{4} \geq \frac{z-1+^{z-1} c_{2}}{q 1}-w_{5}$. where $\mathrm{q}_{1}$ is the probability of getting the inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^{2}-\left(\frac{\varphi(S)}{2}\right)^{2}$

## VI. APPLICATION OF EULER'S TOTIENT FUNCTION AND NUMBER THEORETIC FUNCTIONS IN STUDENT'S PENCIL COMPASS

Mathematical instruments are used to understand mathematical constructions and concepts . If we talk about mathematical instruments then there are many mathematical instruments like protractor, ruler, set-square, divider, pencil compass and etc . There are two types of mathematical instruments, one is used by student and other are used by teachers. Mathematical instruments used by teachers are very large as compare to mathematical instruments used by student because they are used in black boards or white boards. The mathematical instruments used by student are smaller because they are usable in their books only. If we talk about the student's pencil compass, then one can form the circles with a radius of about 1 to 15 centimeters .

Here we have applied the concepts of number theory .
If it is assumed that a circle of radius up to 20 centimeter is being formed by student's pencil compass then also the following result will be correct . Normally circles of radius up to 14 or 15 centimeters are formed by the student's pencil compass .

## VII. RESULT

If we draw all the possible circles by student's pencil compass having radius $=r(r=1,2,3,4, \ldots \ldots$, maximum possible radius to be made by student's pencil compass ), then 6 is the only number whose every positive proper divisors as a radius follow the relation

$$
\varphi(2 r)+\tau(2 \mathrm{r})+\sigma(2 \mathrm{r})=\text { greatest integer function of } \mathrm{P}
$$

and the number 6 (itself) as a radius follow the relation

$$
\begin{aligned}
& \varphi(2 r)+\tau(2 \mathrm{r})+\sigma(2 \mathrm{r})=\text { lowest integer function of } \mathrm{P} . \\
& \varphi(2 r)+\tau(2 \mathrm{r})+\sigma(2 \mathrm{r}) \text { can also be written as } \varphi(d)+\tau(\mathrm{d})+\sigma(\mathrm{d}),
\end{aligned}
$$

where $\mathrm{P}=\pi \times \mathrm{d}, \pi=3.1415$ (four digits after the decimal point), P is circumference(perimeter) of circle and d is diameter of circle .
NOTE: (a) Draw all the possible circles by student's pencil compass means to make circles of each radius which is made by student's pencil compass .

Where $\varphi(d)$ is the is the euler's phi function which denote the number of positive integers not exceeding d that are relatively prime to d .
$\tau(\mathrm{d})$ denote the number of positive divisors of d and $\sigma(\mathrm{d})$ denote the sum of these divisors .
In mathematical form : If $\mathrm{r}=1, \varphi(2)+\tau(2)+\sigma(2)=$ greatest integer function of $2 \times 3.1415 \times 1=6$,
If $r=2, \varphi(4)+\tau(4)+\sigma(4)=$ greatest integer function of $2 \times 3.1415 \times 2=12$,
If $\mathrm{r}=3, \varphi(6)+\tau(6)+\sigma(6)=$ greatest integer function of $2 \times 3.1415 \times 3=18$,
If $\mathrm{r}=6, \varphi(12)+\tau(12)+\sigma(12)=$ lowest integer function of $2 \times 3.1415 \times 6=38$,
Where 1,2 and 3 are the positive proper divisors of 6 .

## VIII. ACKNOWLEDGEMENT

I would like to thanks Dr. Pravin Hudge, Dr. V. Patil, Dr. Vikas Deshmane, and Mandar Khasnis sir for useful comments .

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