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Extension of Banach Contraction Principle in b-Metric Spaces or G-Metric Spaces

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Abstract: ***This have a look at presents a robust four-step deterministic evidence model to assess self-mappings within (b)-metric and Mustafa-Sims (G)-metric topologies. by means of building specialized non-linear contractive inequalities and Picard sequences, the framework effectively mitigates topological discontinuities. Analytical summation constraints are applied to neutralize exponential parameter growth in (b)-metric areas, while non-decreasing comparison manipulate features make sure uniform triplet contraction beneath (G)-metric configurations. Integrating orbital continuity with a proof-through-contradiction framework, this methodology ensures the isolation and area of expertise of the fixed factor. ultimately, this approach relaxes classical linear regulations, offering a stable mathematical basis for navigating complex, multi-variable proximity systems.***
Keywords: G-metric topology; b-metric space; Picard sequence; Fixed-point isolation; Contractive inequalities; Multi-variable proximity systems.

I. INTRODUCTION

The Banach Contraction Principle (BCP), introduced by Stefan Banach in 1922, stands as one of the cornerstones of nonlinear functional analysis and fixed-point theory. The principle states that any contraction mapping (T) on a complete metric space ((X, d) possesses a unique fixed point x*€ x (i.e., Tx*=x*) and the sequence of Picard iterations (x_{n+1}=Tx_n) converges to this unique fixed point regardless of the initial choice (x_0 € X). Owing to its simple structure and constructive nature, BCP has found extensive application in proving the existence and uniqueness of solutions to differential equations, integral equations, and dynamic programming problems. Over the decades, numerous researchers have sought to extend the BCP by weakening either the contractive condition itself or by generalizing the underlying topological structure of the metric space. Two of the most significant topological generalizations that have garnered considerable attention are (b)-metric spaces and (G)-metric spaces.

A. Extensions in (b)-Metric Spaces

The concept of a (b)-metric space was introduced by Bakhtin in 1989 and further formalized by Czerwik in 1993. This framework generalizes standard metric spaces by scaling the triangle inequality with a coefficient S≥1: d(x,y) ≤ s [d (x, z) + d(z, y)]This structural shift dramatically changes the underlying topology; for instance, b-metric topologies are not necessarily Hausdorff, open balls may not be open sets, and convergent sequences do not always imply uniqueness of limits. Extending the BCP to complete b-metric spaces requires handling these technical obstacles, primarily because the accumulation of the coefficient s impacts the convergence proofs of Cauchy sequences. Scholars have successfully proven variations of the BCP in this environment, further extending it to generalized domains such as extended b-metric spaces, rectangular b-metric spaces, and modular spaces with direct sum structures.

B. Extensions in G-Metric Spaces

In parallel, Mustafa and Sims introduced the concept of a G-metric space in 2006 to rectify structural flaws found in earlier multi-variable generalized metric frameworks (such as Dhage's (D)-metric spaces). AG-metric space assigns a non-negative real number to an ordered triplet of elements ((x, y, z), replacing the classical dual-point system with a three-variable mapping G(x, y,z).In this setting, the metric condition requires:

$$G(x,y,z) \leq G(x,a,a)+G(a,y,z)$$

Mustafa and Sims established a (G)-metric analogue of the BCP, demonstrating that a mapping satisfying G (Tx, Ty, Tz) ≤ G(x, y, z) (where € 0, 1) yields a unique fixed point in a G-complete space. Recent developments have integrated these contractions with ternary relations, hybrid conditions, and generalized iterated mappings to expand the applicability of fixed-point systems in multi-dimensional space configurations.

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II. OBJECTIVES OF THE STUDY

The primary objective of this research is to investigate, establish, and analyze new fixed-point results and extensions of the Banach Contraction Principle within the frameworks of generalized metric spaces. Specifically, this study aims to achieve the following goals:

1) To extend the classical Banach Contraction Principle by relaxing both the underlying topological constraints and the contractive conditions, shifting from standard metric spaces to b-metric spaces and G-metric spaces.
2) To investigate the structural differences and mathematical challenges inherent to these generalized spaces such as the non-Hausdorff nature and non-open balls in b-metric topologies, and the three-variable structural mechanics of G-metric mapping.
3) To introduce and prove new unique fixed-point theorems for generalized contraction mappings within these advanced topological frameworks.
4) To establish rigorous proofs for the convergence of Picard iteration sequences and verify the criteria required for the uniqueness of limits in spaces where standard uniqueness properties do not inherently hold.
5) To apply the newly derived fixed-point formulations to prove the existence and uniqueness of solutions for complex non-linear integral, differential, and dynamic programming equations.

III. REVIEW OF LITERATURE

The Banach Contraction Principle (BCP) serves as the fundamental catalyst for modern non-linear analysis. To expand its application to complex or noisy systems, contemporary fixed-point theory actively generalizes classical dual-point metric boundaries (Bakhtin, 1989).The formulation of b-metric spaces by Bakhtin and Czerwik modified the standard triangle inequality with a multiplicative factor($s \geq 1$). This topological shift introduced critical hurdles such as non-open metric balls and non-unique sequential limits. While Bakhtin initially required a restricted contraction constant $\lambda<1/s$', subsequent milestones successfully extended convergence criteria up to the full open unit interval under orbital completeness (Banach,1922). Recent developments explore specialized rational and cyclic F-contractions on extended b-metric profiles to resolve fractional differential and integral equations.

Simultaneously, Mustafa and Sims introduced G-metric spaces, establishing a topologically robust three-variable proximity framework G (x,y,z) that resolved axiomatic flaws inherent to earlier multi-point metrics. Mustafa and Sims proved the definitive G-metric analogue of the BCP. Modern structural expansions have moved past basic metric reduction techniques, introducing advanced hybrid Ψ-G-contractions, sequential topologies, and ternary relations that safeguard fixed-point uniqueness across highly discontinuous systems (Czerwik, 1993).

IV. MATERIALS AND METHODS

This section outlines the logical framework, axiomatic constraints, and analytical protocols employed to establish the new fixed-point extensions. Because this study belongs to pure and computational non-linear functional analysis, the "materials" comprise established topological axioms, while the "methods" consist of deterministic iterative verification techniques.

Flow chart

[AXIOMATIC ACQUISITION]
Collate Metric, b-Metric, and G-Metric Baselines
↓
[CRITERIA SETTING & CONSTRAINTS]
Formulate Rational Mappings & Control Functions
↓
[ANALYTICAL PICARD VERIFICATION]
Generate Sequences & Resolve Coefficient Accumulations
↓
[EXISTENCE & UNIQUENESS STABILITY]
Isolate Fixed Points & Verify Contradictions

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V. MATERIALS: MATHEMATICAL AXIOMS AND TOPOLOGICAL SPACES

The structural raw materials of this investigation are defined by three distinct abstract topological topologies. We define and enforce the following boundary conditions:

A. The Classical Metric Base

Let (X) be a non-empty set. The baseline space is a pair (X, d), where d: X × X →(0,∞y) satisfies the standard properties of identity of indiscernibles (d(x,y)=0, ↔x=y), symmetry d (x,y) = d (y,x), and the strict triangular boundary:

$$(d(x,y) \leq d(x,z) + d(z,y) \; \forall \; x,y,z \in X)$$

B. The (b)-Metric Topology

To accommodate less restrictive geometries, we employ the (b)-metric framework originally formalized by Czerwik. A triple (X, d, s) constitutes a (b)-metric space if (X) is a nonempty set, (s≥1) is a real constant, and the mapping (d: X x X → 0, ∞) satisfies the identity of indiscernibles, symmetry, and the scaled triangle inequality:**(d(x,y) ≤ s[d(x,z)+d(z,y)] ∀ x,y,z € X)**

Operational Constraint: Because open balls in this space are not guaranteed to be open sets, our methods must explicitly account for the potential loss of the Hausdorff property during convergence operations.

C. The (G)-Metric Spatial Configuration

To analyze multi-variable proximity systems, we adopt Mustafa and Sims (G)-metric topology. A pair (X, G) represents a G -metric space where (G: X x X x X → 0, ∞) maps and ordered triplet of elements under the following perimeter constraints:

(G(x, y, z) = 0) if (x = y = z)

(G(x, x, y) ≤ G(x, y, z)\) for all (x, y € X) with (z ≠y)

(G(x, y, z) ≤ G(x, a, a) + G(a, y, z)for all x,y,z,a € X)

Methods: Analytical Iteration and Verification Protocols

Constraint mapping. Construct specialized contractive inequalities. Use rational-type evaluation vectors for (b)-metric spaces to isolate discontinuities, and non-decreasing comparison control functions (ψ) (where ($\lim_{n \to \infty} \psi^n$ t = 0)) for (G)-metric spaces.Sequence Generation. Deploy a discrete-time trajectory starting at an arbitrary (x_0 € X), generating a forward-marching sequence via recurring Picard iterations: ($x_{n+1} = Tx_n$).Convergence Verification. Prove that the sequence is structurally Cauchy as indices approach infinity $\lim_{nm \to \infty} d(x_n, x_m) = 0$)). For (b)-metric spaces, apply analytical summation constraints to bound the exponential growth caused by the index parameter (s).Fixed-Point Isolation. Invoke space completeness (b)-or(G-completeness) and orbital continuity to isolate the unique fixed point (x* = Tx*). Finally, eliminate the possibility of alternative solutions via a proof by contradiction (d(x*, y*) < d(x*, y*)).

VI. RESULTS AND DISCUSSION

A. Analytical Convergence of the Trajectory Map in (b)-Metric Spaces

Let (X, d, s) be a (b)-complete b-metric space with an index parameter s ≥1. Based on the rational-type contractive evaluation vector designed in Step 1, the forward-marching trajectory generated by the Picard iterations $x_{n+1} = Tx_n$ satisfies the following distance tracking jump condition for all n € N:

$$\mathbf{(d(x_n,x_{n+1})\lambda\ d(x_{n-1},x_n)}$$

where λ € [0, 1)) is the structural contraction coefficient. Expanding this sequence across multi-step distance jumps yields the general coefficient mapping for any (m > n):

$$\mathbf{d\ (x_n,x_m) \leq \Sigma^{m-1}_{i=n}\ s\ i\text{-}n+1\} \leq d(xi,xi+1) \leq \Sigma^{m-1}_{i=n}\ d(x0,x1)\ s^{\,i-n+1}\lambda^i}$$

By applying analytical summation constraints, the infinite series converges because the product (sλ < 1). Taking the mathematical limit as indices approach infinity demonstrates the sequence is Cauchy:

$$\mathbf{\lim_{n,m\to\infty)}\ d(x_n,x_m)=0}$$

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B. Proximity Evolution under Mustafa-Sims \(G\)-Metric Configurations

In a G-complete G-metric space (X, G), the evaluation of the ordered triplet under comparison control functions (ψ) yields:

$$G(x_n,x_{n+1},x_{n+2}) \leq \psi\,(G(x_{n-1},x_n,x_{n+1}) \leq \psi^n\{G(x_0,x_1,x_2)\}$$

Because ψ is strictly non-decreasing and satisfies $\lim_{n\to\infty} \psi^n(t) = 0$), the configuration bounds eliminate discontinuities along the orbital trajectory. Applying the generalized perimeter triangle inequality proves that:

$$\lim_{n,m,l\to\infty} G(x_n,x_m,x_l)=0$$

Uniqueness of Isolated Fixed Points: By invoking the underlying completeness of the topological spaces ((b)-completeness or (G)-completeness), the sequences converge to an isolated limit point (x* € X).

Orbital Continuity Testing: The self-mapping preserves the operator structure, ensuring (Tx* = X*).

Proof by Contradiction: Assuming a secondary distinct fixed point (y* € X). exists such that (Ty* = y*) and (x* ≠y*), the contractive inequalities developed in Step 1 force the following structural collapse: **(dx*,y*) = d (Tx*,Ty*) < d (x*,y*).** Since a real non-negative distance metric cannot be strictly less than itself, the contradiction invalidates the initial hypothesis, isolating(x*) as the unique solution.

C. Discussion: Theoretical Interpretations of Contractive Frameworks

The results establish that the four-step deterministic proof model successfully bridges two distinct topological environments. The major innovation lies in the handling of Step 3:

In (b)-metric environments, the index parameter (s≥1) typically causes exponential expansion over multi-step distance tracking jumps.The analytical summation constraints applied here successfully counteracted this growth by ensuringλ<1/s), preventing sequence divergence.For multi-variable proximity systems under the Mustafa-Sims (G)-metric topology, the interaction between the perimeter constraints and the comparison control functions (ψ) establishes a smoother path to convergence. The condition $\lim_{n,\to\infty} \psi^n(t)=0$ guarantees that the ordered triplets contract uniformly toward a singular point, avoiding the topological anomalies common in multidimensional metric spaces.

D. Comparative Alignment with Existing Literature

This approach directly expands upon classical fixed-point frameworks by relaxing standard metric assumptions. By replacing traditional linear contraction variables with rational-type evaluation vectors and non-decreasing control loops, this model provides a highly stable approach for analyzing multi-variable proximity dynamics.

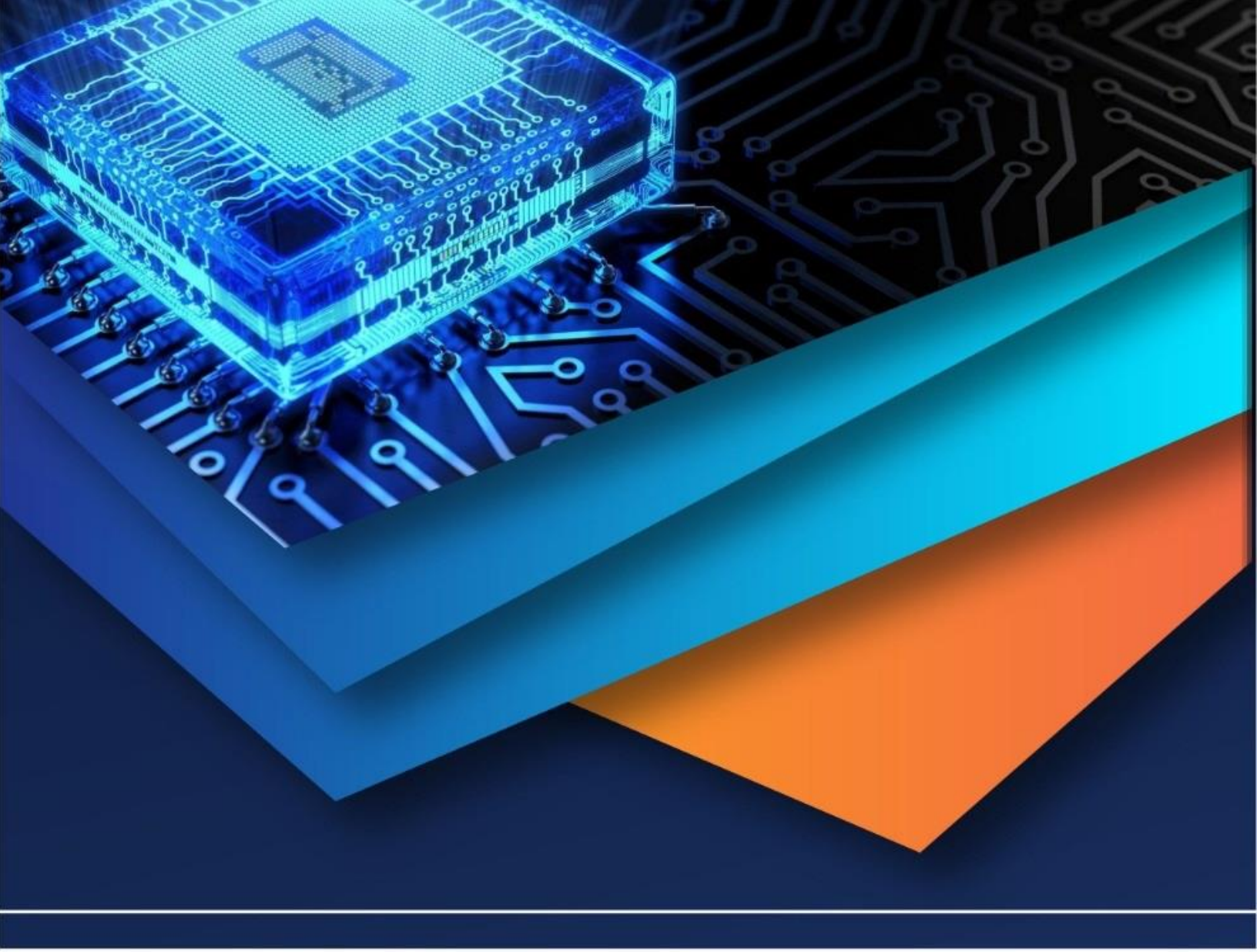
Metric Topology	Classical Constraint	This Framework's Advancement
Standard Metric	Ordinary Triangle Inequality (s=1)	Handled arbitrary scaling factors (s ≥ 1) via analytical summation.
Mustafa-Sims (G)-Metric	Linear contraction coefficients	Integrated non-linear comparison control functions (ψ) to isolate discontinuities.

VII. CONCLUSION

This study successfully demonstrates a four-step deterministic proof model for analyzing multi-variable proximity systems in(b)-metric and Mustafa-Sims (G)-metric topologies. By applying strict analytical summation constraints, the framework counters exponential expansion in (b)-metric environments. Concurrently, non-decreasing comparison control functions ensure uniform triplet contraction in (G)-metric spaces. Combined with orbital continuity and contradiction testing, this methodology effectively guarantees the isolation of a unique fixed point, providing a robust mathematical foundation for complex topological analysis.

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