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A Generalized Subclass of p-VALENT Analytic Functions

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Abstract: In this paper, a class $\Sigma_{p,\alpha}(\alpha, \beta, \mu, k)$ of functions F analytic in the unit disk $U=\{z:|z|<1\}$ of the form

$$F(z) = z^p - \sum_{t=n}^{\infty} a_{p+t} z^{p+t}, (p, n \in \mathbb{N}, a_{p+t} \geq 0 \text{ for } t \geq n) \text{ is considered. Coefficient inequality, distortion theorem, extreme points, starlikeness and convexity for this class are obtained.}$$

Keywords: Analytic functions, p-valent functions, extreme points, radii of starlikeness and convexity.

2010 AMS Subject classification: Primary 30C45.

I. INTRODUCTION

Let $\Delta_{p,n}$ denotes the class of p-valent analytic functions in the unit disk $U=\{z:|z|<1\}$ which are of the form

$$(1.1) \quad F(z) = z^p - \sum_{t=n}^{\infty} a_{p+t} z^{p+t}, p, n \in \mathbb{N}, a_{p+t} \geq 0.$$

A function $P(z) = p + p_n z^n + \dots, (n \geq 1)$ analytic in U is said to be in $M_{p,n}(\alpha)$ if

$$(1.2) \quad |P(z) - p| < (p - \alpha), \quad 0 \leq \alpha < p, \quad z \in U.$$

Alternatively, in terms of subordination, it is said that $P(z)$ is in $M_{p,n}(\alpha)$ if

$$(1.3) \quad P(z) \prec p + (p - \alpha)z$$

where ‘ \prec ’ stands for subordination which is defined by saying that F is subordinate to G written as $F \prec G$ if $F(z) = G(\phi(z)), z \in U$ for some analytic functions $\phi(z)$ such that

$$\phi(0) = 0 \quad \text{and} \quad |\phi(z)| < 1 \quad \text{for } z \in U.$$

Now, a class criterion $M_{p,n}(\alpha, \beta, \mu, k)$ whose members $P(z) = p + p_n z^n \dots (n \geq 1)$, analytic in U satisfy the condition

$$(1.4) \quad P(z) \prec \frac{k + (k - 2\mu\alpha)\beta z}{1 - (2\mu - 1)\beta z}$$

with $1 \leq 2\mu \leq 2, 2\mu\alpha < k \leq p, 0 < \beta \leq 1$ and $0 \leq \alpha < \frac{p}{2\mu} - \frac{(p-k)}{\mu}$ is considered.

Equivalently (1.4) can be written as:

$$(1.5) \quad \left| \frac{P(z) - k}{(2\mu - 1)P(z) + (k - 2\mu\alpha)} \right| < \beta$$

Note that $M_{p,n}(\alpha, 1, 1/2, p) \equiv M_{p,n}(\alpha)$ and $M_{p,n}(\alpha, \beta, \mu, p) \equiv M_{p,n}(\alpha, \beta, \mu)$.

On putting $P(z) = \frac{F'(z)}{z^{p-1}}$ in (1.5) it follows that

$$(1.6) \quad \left| \frac{\frac{F'(z)}{z^{p-1}} - k}{(2\mu - 1)\frac{F'(z)}{z^{p-1}} + (k - 2\mu\alpha)} \right| < \beta, \quad F(z) \in \Delta_{p,n}.$$

The class of such functions $F(z)$ satisfying (1.6) is denoted by $\Sigma_{p,\alpha}(\alpha, \beta, \mu, k)$ and

$$\Sigma_{p,n}(\alpha, \beta, \mu, p) \equiv \Sigma_{p,n}(\alpha, \beta, \mu).$$

Clearly, these classes are more general than the classes studied by Gupta and Jain [2,3], Aouf [1], Juneja–Mogra [4], Thirupathi Reddy [6] and Kulkarni, Aouf, Joshi [5] etc.

In this paper, coefficient inequalities, distortion theorem, closure theorem, radii of starlikeness and convexity for the class $\Sigma_{p,n}(\alpha, \beta, \mu, k)$ are obtained.

II. COEFFICIENT INEQUALITIES:

Theorem 2.1: A function $F(z) \in T_{p,n}$ is in the class $\Sigma_{p,n}(\alpha, \beta, \mu, k)$ if and only if

$$(2.1) \quad \sum_{t=n}^{\infty} (p+t)a_{p+t} \leq \frac{\{2\mu\beta(p-\alpha) - (p-k)(1+\beta)\}}{1 + (2\mu - 1)\beta}.$$

The result is sharp, the extremal function being

$$(2.2) \quad F(z) = z^p - \frac{\{2\mu\beta(p-\alpha) - (p-k)(1+\beta)\}}{(p+t)\{1 + (2\mu - 1)\beta\}} z^{p+t} \quad \text{for } t \geq n.$$

Proof: Let $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$, then

$$\begin{aligned} & \left| \frac{\frac{F'(z)}{z^{p-1}} - k}{(2\mu - 1)\frac{F'(z)}{z^{p-1}} + (k - 2\mu\alpha)} \right| \\ &= \left| \frac{(p-k) - \sum_{t=n}^{\infty} (p+t)a_{p+t}z^t}{\{(2\mu - 1)p + (k - 2\mu\alpha)\} - (2\mu - 1)\sum_{t=n}^{\infty} (p+t)z^t} \right| < \beta. \end{aligned}$$

On letting $z \rightarrow 1^-$ through real values, it gives

$$\begin{aligned} & (p-k) + \sum_{t=n}^{\infty} (p+t)a_{p+t} \\ & \leq \beta\{(2\mu - 1)p + (k - 2\mu\alpha)\} - \beta(2\mu - 1)\sum_{t=n}^{\infty} (p+t)a_{p+t} \end{aligned}$$

or,

$$\sum_{t=n}^{\infty} (p+t)a_{p+t} \leq \frac{2\mu\beta(p-\alpha) - (p-k)(1+\beta)}{1 + (2\mu - 1)\beta}.$$

Conversely, let (2.1) holds, then for $|z| = 1$

$$|F'(z)z^{1-p} - k| - \beta |(2\mu - 1)F'(z)z^{1-p} + (k - 2\mu\alpha)|$$

$$\begin{aligned} &\leq (p - k) + \sum_{t=n}^{\infty} (p + t)a_{p+t} \\ &\quad - \beta\{2\mu - 1\}p + (k - 2\mu\alpha) + \beta(2\mu - 1)\sum_{t=n}^{\infty} (p + t)a_{p+t} \\ &\leq \{1 + \{(2\mu - 1)\beta\}\sum_{t=n}^{\infty} (p + t)a_{p+t} - 2\mu\beta(p - \alpha) + (p - k)(1 + \beta)\} \\ &\leq 0. \end{aligned}$$

The result is sharp for function given by (2.2).

Corollary 2.2: For $k = p$, $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, p)$ if and only if

$$\sum_{t=n}^{\infty} (p + t)a_{p+t} \leq \frac{2\mu\beta(p - \alpha)}{1 + \beta(2\mu - 1)}, \quad 0 \leq \alpha < \frac{p}{2\mu},$$

$$0 < \beta \leq 1, 1 \leq 2\mu \leq 2.$$

For $n = 1$, the result of Kulkarni et.al. [5] follows.

Corollary 2.3: For $k = p$ and $\mu = 1$, $F(z) \in \Sigma_{p,n}(\alpha, \beta, 1, p)$ if and only if

$$\sum_{t=n}^{\infty} (p + t)a_{p+t} \leq \frac{2\beta(p - \alpha)}{1 + \beta}, \quad 0 \leq \alpha < p, 0 < \beta \leq 1.$$

For $n = 1$, $\beta = 1$, the result of Juneja and Mogra [4] follows.

For $n = 1$, $p = 1$, the result of Gupta and Jain [3] follows.

Corollary 2.4: For $k = p$, $\mu = \frac{1 + \nu}{2}$, $F(z) \in \Sigma_{p,n}\left(\alpha, \beta, \frac{1 + \nu}{2}, p\right)$, if and only if

$$\sum_{t=n}^{\infty} (p + t)a_{p+t} \leq \frac{(1 + \nu)\beta(p - \alpha)}{1 + \beta\nu}, \quad 0 \leq \alpha < \frac{p}{2}.$$

For $n = 1$, the result of Thirupathi Reddy [6] follows.

III. DISTORTION THEOREM

Theorem 3.1: If $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$, then for $|z| = r < 1$

$$\begin{aligned} (3.1) \quad &r^p - \left[\frac{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)}{(p + n)\{1 + \beta(2\mu - 1)\}} \right] r^{p+n} < |F(z)| \\ &< r^p + \left[\frac{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)}{(p + n)\{1 + \beta(2\mu - 1)\}} \right] r^{p+n}. \end{aligned}$$

and

$$\begin{aligned} (3.2) \quad &pr^{p-1} - \left[\frac{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)}{\{1 + \beta(2\mu - 1)\}} \right] r^{p+n-1} < |F'(z)| \\ &< pr^{p-1} + \left[\frac{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)}{\{1 + \beta(2\mu - 1)\}} \right] r^{p+n-1}. \end{aligned}$$

Proof: From Theorem 2.1, it follows that

$$(3.3) \quad \sum_{t=n}^{\infty} a_{p+t} < \left[\frac{2\mu\beta(p-\alpha) - (p-k)(1+\beta)}{(p+n)\{1+\beta(2\mu-1)\}} \right].$$

Hence

$$\begin{aligned} |F(z)| &< r^p + \sum_{t=n}^{\infty} a_{p+t} r^{p+t} \\ &< r^p + \left[\frac{2\mu\beta(p-\alpha) - (p-k)(1+\beta)}{(p+n)\{1+\beta(2\mu-1)\}} \right] r^{p+n} \end{aligned}$$

and

$$\begin{aligned} |F(z)| &> r^p - \sum_{t=n}^{\infty} a_{p+t} r^{p+t} \\ &> r^p - \left[\frac{2\mu\beta(p-\alpha) - (p-k)(1+\beta)}{(p+n)\{1+\beta(2\mu-1)\}} \right] r^{p+n}. \end{aligned}$$

Hence (3.1) follows.

In the same way, it follows that

$$\begin{aligned} |F'(z)| &< pr^{p-1} + \sum_{t=n}^{\infty} (p+t)a_{p+t} r^{p+t-1} \\ &< pr^{p-1} + \left[\frac{2\mu\beta(p-\alpha) - (p-k)(1+\beta)}{\{1+\beta(2\mu-1)\}} \right] r^{p+n-1} \end{aligned}$$

and

$$\begin{aligned} |F'(z)| &> pr^{p-1} - \sum_{t=n}^{\infty} (p+t)a_{p+t} r^{p+t-1} \\ &> pr^{p-1} - \left[\frac{2\mu\beta(p-\alpha) - (p-k)(1+\beta)}{\{1+\beta(2\mu-1)\}} \right] r^{p+n-1}. \end{aligned}$$

This completes the proof of the theorem.

The above bounds are sharp. Equalities can be attained for the function

$$(3.4) \quad F(z) = z^p - \left[\frac{2\mu\beta(p-\alpha) - (p-k)(1+\beta)}{(p+n)\{1+\beta(2\mu-1)\}} \right] z^{p+n}, \quad z = \pm r.$$

IV. CLOSURE THEOREMS

Theorem 4.1: If $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$ and $G(z) = z^p - \sum_{t=n}^{\infty} b_{p+t} z^{p+t}$ are also in

$\Sigma_{p,n}(\alpha, \beta, \mu, k)$, then $H(z) = z^p - \frac{1}{2} \sum_{t=n}^{\infty} (a_{p+t} + b_{p+t}) z^{p+t}$ is also in $\Sigma_{p,n}(\alpha, \beta, \mu, k)$.

Proof: Since $F(z)$ and $G(z)$ both belong to $\Sigma_{p,n}(\alpha, \beta, \mu, k)$, then from Theorem 2.1

$$(4.1) \quad \{1 + \beta(2\mu - 1)\} \sum_{t=n}^{\infty} (p+t)a_{p+t} \leq 2\mu\beta(p - \alpha) - (p - k)(1 + \beta).$$

and

$$(4.2) \quad \{1 + \beta(2\mu - 1)\} \sum_{t=n}^{\infty} (p+t)b_{p+t} \leq 2\mu\beta(p - \alpha) - (p - k)(1 + \beta).$$

So for $H(z)$, it follows that

$$\begin{aligned} \frac{1}{2} \{1 + \beta(2\mu - 1)\} \sum_{t=n}^{\infty} (p+t)(a_{p+t} + b_{p+t}) \\ \leq 2\mu\beta(p - \alpha) - (p - k)(1 + \beta). \end{aligned}$$

Using (4.1) and (4.2). Therefore, $H(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$.

V. EXTREME POINTS

Theorem 5.1: Let $F_{n-1}(z) = z^p$ and

$$F_t(z) = z^p - \left[\frac{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)}{(p+t)\{1 + \beta(2\mu - 1)\}} \right] z^{p+t}, \quad t \geq n \text{ then } F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k) \text{ if and only if it can be expressed in}$$

the form

$$F(z) = \sum_{t=n-1}^{\infty} \lambda_t F_t(z), \text{ where } \lambda_t \geq 0 \text{ and } \sum_{t=n-1}^{\infty} \lambda_t = 1.$$

Proof: Suppose

$$F(z) = \sum_{t=n-1}^{\infty} \lambda_t F_t(z) = z^p - \sum_{t=n}^{\infty} \left[\frac{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)}{(p+t)\{1 + \beta(2\mu - 1)\}} \right] \lambda_t z^{p+t},$$

then

$$\begin{aligned} \sum_{t=n}^{\infty} (p+t) \frac{\{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)\} \lambda_t}{(p+t)\{1 + \beta(2\mu - 1)\}} \\ \leq \frac{\{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)\}}{1 + \beta(2\mu - 1)}. \end{aligned}$$

Thus, by Theorem 2.1, $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$.

Conversely, suppose $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$. Hence, by Theorem 2.1, it follows that

$$a_{p+t} \leq \frac{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)}{(p+t)\{1 + \beta(2\mu - 1)\}}, \quad t \geq n$$

Setting

$$\lambda_n = \frac{(p+n)\{1 + (2\mu - 1)\beta\}}{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)} a_{p+n}, \quad n = 1, 2, \dots$$

and

$$\lambda_{n-1} = 1 - \sum_{t=n}^{\infty} \lambda_t.$$

It follows that

$$F(z) = \sum_{t=n-1}^{\infty} \lambda_t F_t(z).$$

This completes the proof.

The extreme points for the class $\Sigma_{p,n}(\alpha, \beta, \mu, k)$ are given by

$$F_{n-1}(z) = z^p$$

and

$$F_t(z) = z^p - \left[\frac{2\mu\beta(p-\alpha) - (p-k)(1+\beta)}{(p+t)\{1+\beta(2\mu-1)\}} \right] z^{p+t}, \quad t \geq n.$$

VI. RADIUS OF STARLIKENESS:

Theorem 6.1: If $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$, then the function $F(z)$ is starlike in the disk

$0 < |z| < r = r(\alpha, \beta, \mu, k, n)$ where

$$r(\alpha, \beta, \mu, k, n) < \inf_{n \in \mathbb{N}} \left\{ \frac{p\{1 + (2\mu - 1)\beta\}}{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)} \right\}^{1/n}, \quad n \in \mathbb{N}$$

Proof: It is enough to show that

$$\left| \frac{zF'(z)}{F(z)} - p \right| < p \text{ for } |z| < 1.$$

or,

$$\left| \frac{zF'(z)}{F(z)} - p \right| = \left| \frac{-\sum_{t=n}^{\infty} t a_{p+t} z^t}{1 - \sum_{t=n}^{\infty} a_{p+t} z^t} \right| < p$$

or,

$$\sum_{t=n}^{\infty} t a_{p+t} |z|^t < p \left[1 - \sum_{t=n}^{\infty} a_{p+t} |z|^t \right]$$

or,

$$\sum_{t=n}^{\infty} \left(\frac{p+t}{p} \right) a_{p+t} |z|^t < 1.$$

But, Theorem 2.1 gives

$$\sum_{t=n}^{\infty} (p+t) a_{p+t} < \frac{2\mu\beta(p-\alpha) - (p-k)(1+\beta)}{1+\beta(2\mu-1)}.$$

Thus, $F(z)$ is starlike if

$$|z| < \left\{ \frac{p[1 + \beta(2\mu - 1)]}{2\mu\beta(p - \alpha) - (p - k)(1 + \beta)} \right\}^{1/n}, \quad n = 1, 2, \dots$$

VII. RADIUS OF CONVEXITY

Theorem 7.1: If $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$, then $F(z)$ is convex in the disk $0 < |z| < r = r(\alpha, \beta, \mu, k, n)$, where

$$r(\alpha, \beta, \mu, k, n) < \inf_{n \in \mathbb{N}} \left\{ \frac{p^2 \{1 + (2\mu - 1)\beta\}}{(p + n)[2\mu\beta(p - \alpha) - (p - k)(1 + \beta)]} \right\}^{1/n} \quad n = 1, 2, \dots$$

Proof: Putting $zF'(z)$ in place of $F(z)$ in Theorem 6.1, the result follows.

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