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Holistic Solutions of Cubic Diophantine Equation with Four unknowns $x^3+y^3+2(x-y)^2(x+y)=20zp^2$

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Abstract: The non-homogenous cubic Diophantine equation with four unknowns represented by $x^3+y^3+2(x-y)^2(x+y)=20zp^2$ is analysed for its pattern of stringifies integral solutions. A few properties among the solutions and some special polygonal numbers are presented.
Keywords: Cubic Diophantine equation, integral solution, polygonal number, gnomonic number, square number, star number.

I. INTRODUCTION

Science is the universal language of the world, imparting knowledge about numbers, patterns, formulas, and forms. Since all other branches rely on number theory for their final outcomes, studying number theory is crucial. The study of number theory is a broad and intriguing area of Mathematics. The French Mathematician Pierre Fermat is credited with coining the term "number theory" in the seventeenth century. In the branch of pure Mathematics known as number theory, integers and integral valued functions are studied. Focused on a broad range of problems that stem from the study of numbers and specifically on the characteristics of integers.

Since almost two millennia ago, number theory has captivated and motivated Mathematicians as well as laypeople. Developed through the tireless efforts of Mathematicians worldwide, it is a reliable and essential body of knowledge. It's interesting to note that number theorists have fortunately developed some of the most advanced mathematical instruments ever and are still pushing the boundaries of knowledge. Number theory is an ancient subject with a wide range of applications, making it by its very nature a science that requires a high degree of rigor. Numerous topics may be distinguished within number theory based on the questions that are studied and the methods that are employed. A Diophantine equation is a polynomial equation that involves at least two unknowns and only has integer solutions. One of the earliest people to apply symbolism to algebra, Diophantus of Alexandria, a 3rd-century Hellenistic Mathematician, explored these equations and is the subject of the term "Diophantine." It is shown that two chemistry-related problems can be resolved using the diophantine equations Mathematical method: determining the molecular formula of a substance and balancing chemical equations. There are various dissimilar ternary cubic diophantine equations. To learn the fundamentals of number theory refer [1-3]. To understand it in more exhaustive [4-6] is noticeable. For this inherent decrypt to ternary cubic diophantine equations [7-15]is observable. This article discuss about ternary cubic diophantine equation $x^3+y^3+2(x-y)^2(x+y)=20zp^2$ with four unknowns for their non-trivial integral solutions in three different patterns. Also connections between the solutions and polygonal, square, gnomonic, star numbers are discussed.

Notations

CS_n = Centred square number of rank n.

$\Pr_n$ = Pronic number of rank n.

Gno_n = Gnomonic number of rank n.

CH_n = Centred Hexagonal number of rank n.

$T_{m,n}$ = Polygonal number of rank n with size m.

$Star_n$ = Star number of rank n.

II. BASIC DEFINITIONS

1) *Definition 1.1*

A Diophantine equation is a polynomial equation, usually involving two or more unknowns, such that the only solutions of interest are the integer ones.

2) *Definition 1.2*

If an integer $m \neq 0$ and $m/a-b$, we say that a is congruent to b modulo m and write

$$a \equiv b(\mathrm{mod}\, m)$$

3) *Definition 1.3*

A Perfect square is a number that can be expressed as k^2, where k is an integer.

Example:

$10^2 = 100$

4) *Definition 1.4*

A Nasty number is a non-negative integer with atleast four distinct factors such that the difference between one pair of factors is equal to the addition of the another pair and the multiplication of each pair is equal to the number.

Example:

6 is a nasty number.

$6 = 6 \times 1 = 2 \times 3$ and $6 - 1 = 2 + 3$

5) *Definition 1.5*

A Duck number is a number which has zeros present in it, but there should be no zero present in the beginning of the number.

Example:

3210, 7056, 8430, 709 are duck numbers

083, 0430 are not duck numbers

6) *Definition 1.6*

The Happy couple number for which the sum of the squares of the digits eventually equals 1.

Example:

79 is a happy couple number because

$7^2 + 9^2 = 130;\ 1^2 + 3^2 + 0^2 = 10;\ 1^2 + 0^2 = 1$

7) *Definition 1.7*

An Abundant number is a positive integer for which the sum of its proper divisor is greater than the number.

Example:

12 is abundant number.

8) *Definition 1.8*

A composite numbers are the numbers which have more than two factors.

Example:

561 and 4955 are composite numbers.

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III. METHOD OF ANALYSIS

The cubic Diophantine equation with four unknowns to be considered for its integral solution is

$$x^3 + y^3 + 2(x-y)^2(x+y) = 20zp^2 \tag{1}$$

After using the linear transformation

$$x = u+v \quad , y = u-v \ , z = 2u \tag{2}$$

where u and v are non-zero parameters, eqn (1) leads to

$$u^2 + 11v^2 = 20p^2 \tag{3}$$

The following list offers three different methods of solving (3).The appropriate values of x and y are derived from the values of u and v by using (2).

A. *Pattern I*

Assume that

$$p = p(a,b) = a^2 + 11b^2 \tag{4}$$

Write, $20 = (3+i\sqrt{11})(3-i\sqrt{11})$ (5)

Substitute (4) and (5) in (3), we get

$$u^2 + 11v^2 = (3+i\sqrt{11})(3-i\sqrt{11})(a^2+11b^2)$$

By the method of factorization the above equation can be written as

$$(u+i\sqrt{11}v)(u-i\sqrt{11}v) = (3+i\sqrt{11})(3-i\sqrt{11})(a+i\sqrt{11}b)^2(a-i\sqrt{11}b)^2$$

On comparing the like terms

$$(u+i\sqrt{11}v) = (3+i\sqrt{11})(a+i\sqrt{11}b)^2 \tag{6}$$

On comparing real and imaginary parts from the above equation we get

$$\left.\begin{aligned} u &= 3a^2 - 22ab - 33b^2 \\ v &= a^2 + 6ab - 11b^2 \\ z &= 6a^2 - 44ab - 66b^2 \end{aligned}\right\} \tag{7}$$

Using (7) in (2) the corresponding non-zero distinct integer solution are found to be

$$x = x(a,b) = 4a^2 - 16ab - 44b^2$$
$$y = y(a,b) = 2a^2 - 28ab - 22b^2$$
$$z = z(a,b) = 6a^2 - 44ab - 66b^2$$
$$p = p(a,b) = a^2 + 11b^2$$

Properties

1. $x(a,1) + y(a,1) + z(a,1) - 12\Pr_a + 50Gn_a \equiv 0 (\mathrm{mod}\ 50)$
2. $x(a,1) - y(a,1) - z(a,1) - CS_a + 2T_{8,a} - 27Gn_a \equiv 0 (\mathrm{mod}\ 70)$
3. $x(a,1) + y(a,1) + z(a,1) + p(a,1) - T2_8, a + 38Gn_a \equiv 0 (\mathrm{mod}\ 38)$
4. $x(a,1) + y(a,1)$ is duck number.
5. $8x(1,1) - y(1,1)$ is square number.

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B. Pattern II

20can also be written as

$$20 = \frac{\left(13 + i\sqrt{11}\right)\left(13 - i\sqrt{11}\right)}{9} \tag{8}$$

Substitute (4) and (8) in (3) and by a method of factorization

$$\left(u + i\sqrt{11}v\right)\left(u - i\sqrt{11}v\right) = \frac{\left(13 + i\sqrt{11}\right)\left(13 - i\sqrt{11}\right)}{9}\left(a + i\sqrt{11}b\right)^2 \left(a - i\sqrt{11}b\right)^2$$

On comparing thelike terms, we get

$$\left(u + i\sqrt{11}v\right) = \frac{\left(13 + i\sqrt{11}\right)\left(a + i\sqrt{11}b\right)^2}{3} \tag{9}$$

$$\left(u + i\sqrt{11}v\right) = \frac{1}{3}\left(13a^2 + i26ab\sqrt{11} - 143b^2 + ia^2\sqrt{11} - 22ab - i11b^2\sqrt{11}\right)$$

On comparing real and imaginary parts from the above equation,we get

$$\left.\begin{aligned} u &= \frac{1}{3}\left(13a^2 - 143b^2 - 22ab\right) \\ v &= \frac{1}{3}\left(a^2 + 26ab - 11b^2\right) \\ p &= a^2 + 11b^2 \end{aligned}\right\} \tag{10}$$

Since our aim is to find integral solutions, so that take $a = 3A$ and $b = 3B$ in (10), the values of u and v are

$$\left.\begin{aligned} u &= 39a^2 - 429b^2 - 66ab \\ v &= 3a^2 + 78ab - 33b^2 \\ p &= a^2 + 11b^2 \end{aligned}\right\} \tag{11}$$

The corresponding non-zero distinct integer solution of equation are found to be

$x = x(A, B) = 42A^2 + 12AB - 462B^2$

$y = y(A, B) = 36A^2 - 144AB - 396B^2$

$z = z(A, B) = 78A^2 - 132AB - 858B^2$

$p = p(A, B) = 9a^2 + 99b^2$

Properties

1. $x(a,1) - y(a,1) - z(a,1) - p(a,1) + 18T_{11}, a - 113Gn_a + a \equiv 0 \pmod{113}$
2. $x(a,1) + y(a,1) + z(a,1) + p(a,1) - 83CS_a + 2T_{3,a} + 49Gn_a + a \equiv 0 \pmod{1583}$
3. $x(a,1) - y(a,1) - z(a,1) + 24CH_a - 108Gn_a \equiv 0 \pmod{924}$
4. $x(1,1) + y(1,1)$ is abundant number.

5. $12z(1,1)$ is square number.

C. Pattern III

Write, $20 = \dfrac{(18+i4\sqrt{11})\;(18-i\sqrt{11})}{25}$ (12)

Substitute (4) and (12) in (3), and by the method of factorization,

$$\left(u+i\sqrt{11}v\right)\left(u-i\sqrt{11}v\right) = \frac{(18+i4\sqrt{11})\,(18-i\sqrt{11})}{25}\;(a+i\sqrt{11})^2\;(a-i\sqrt{11}b)^2$$

On comparing like terms, we get

$$\left(u+i\sqrt{11}v\right) = \frac{\left(18+i4\sqrt{11}\right)}{5}\left(a+i\sqrt{11}b\right)^2 \quad (13)$$

On equating real and imaginary parts, leads to

$$\left.\begin{aligned} u &= \frac{1}{5}\left(18a^2 - 88ab - 198b^2\right) \\ v &= \frac{1}{5}\left(4a^2 + 36ab - 44b^2\right) \\ p &= a^2 + 11b^2 \end{aligned}\right\} \quad (14)$$

Since our aim is to find integral solutions, so that take
$a = 5A$ and $b = 5B$ in (14), the values of u and v are

$$\left.\begin{aligned} u &= 90A^2 - 990B^2 - 440AB \\ v &= 20A^2 - 220B^2 + 180AB \end{aligned}\right\}$$

The corresponding non-zero distinct integer solution of equation are found to be

$$x = x(a,b) = 110a^2 - 260ab - 1210b^2$$
$$y = y(a,b) = 70a^2 - 620ab - 770b^2$$
$$z = z(a,b) = 180a^2 - 880ab - 1980b^2$$
$$p = p(a,b) = a^2 + 11b^2$$

Properties

1. $x(a,1) + y(a,1) + z(a,1) + p(a,1) - 65Star_a + T_{7,a} + 685Gn_a + 3a \equiv 0(\mathrm{mod}\ 4435)$
2. $x(a,1) + y(a,1) + z(a,1) - 60Star_a + 850Gn_a + 3a \equiv 0(\mathrm{mod}\ 4260)$
3. $x(a,1) - y(a,1) - z(a,1) - p(a,1) - 55CH_a - 539Gn_a + 3a \equiv 0(\mathrm{mod}\ 1839)$
4. $x(1,1) - y(1,1) - z(1,1) + p(1,1)$ is abundant number.
5. $3p(1,1)$ is duck number.

IV. CONCLUSION

In this analysis, the cubic diophantine equation $x^3 + y^3 + 2(x-y)^2(x+y) = 20zp^2$ for its non-zero distinct integral solutions in three different patterns are explored and few interesting properties are determined. To conclude, one may search for solving the cubic problem under consideration as well as cubic diophantine equation with many variables.

REFERENCES

[1] Dickson L.E, " History of the theory of numbers", Chelsia Publishing Co, Sol II, New York 1952.

[2] Cira, Octavian, and FlorentinSmarandache. Solving Diophantine Equations. Infinite Study, 2014.

[3] Carmichael.R.D, "The theory of Numbers and Diophantine Analysis", Dover Publications, New York,1959.

[4] JanakiDr G. " Explorationsin Diophantine Equations", B P International, 2023.

[5] Telangs S.G, " Number Theory ,Tata McGraw-Hill Publishing Company, New Delhi 1996

[6] Janaki, G and P. Saranya. "On the ternary Cubic Diophantine equation $5\left(x^2+y^2\right)-6xy+4\left(x+y\right)+4=40z^3$ International Journal of Science and Research-online 5.3 (2016): 227-229.

[7] Janaki, G and M. Krishnaveni "On the Integer Solutions of the Homogeneous Biquadratic Diophantine Equation." International Journal of Engineering Science 7275 (2016).

[8] Janaki, G and P. Saranya "On the Ternary Cubic Diophantine Equation " International Journal of Science and Research-Online 5.3 (2016): 227-229.

[9] Janaki G, Saranya C, ''Integral solutions of the Ternary Cubic Equation $3\left(x^2+y^2\right)-4xy+2\left(x+y+1\right)=972z^3$ ", International Journal for Research in Applied science and Engineering and Technology vol-4, Issues 3, Page No: 665-669, March 2017.

[10] Janaki G and C. Saranya. "Integral Solutions of the Non-Homogeneous Sixtic Equation with three unknown $3\left(x^2+y^2\right)-2xy=972z^6$." Int. J. Sci. Res. in Multidisciplinary Studies Volume- 6.3 (2020).

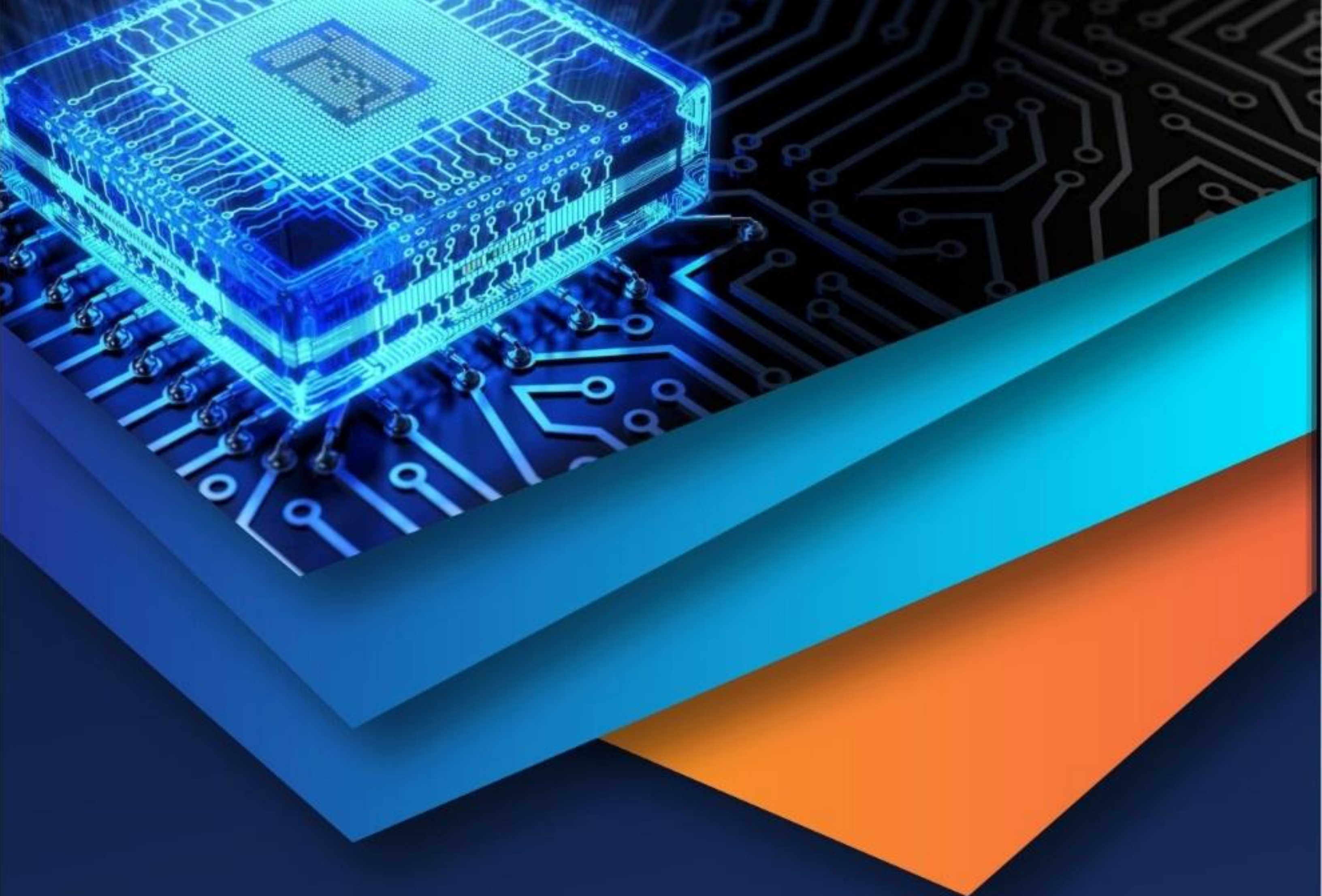
[11] Janaki G, GowriShankari A , ''Properties of the Ternary Cubic Equation $5x^3-3y^2=z^3$ '',International Journal for Research in Applied Science and Engineering Technology , volume 10, Issues III, Page No: 231-234, August 2022.

[12] Saranya C and P. Kayathri "Integral Solutions of the Ternary Cubic Equation $6\left(x^2+y^2\right)-11xy=288z^3$ ", International Journal for Research in Applied Science and Engineering Technology , volume-10, Issues I, January 2022

[13] Vidhyalakshmi S, Gopalan MA, "On Finding Integer Solutions to Non-homogeneous Ternary Cubic Equation $x^2+xy+y^2=\left(m^2+3n^2\right)z^3$", Journal of Advanced Education and Sciences 2 (4) 28-31, 2022.

[14] Janaki G, and M. Aruna "On Integer Solutions of the Ternary Quadratic Equation $3a^2+3r^2-2ar=332n^2$ ", International Journal for Research in Applied Science and Engineering Technology , volume-11, Issues III, March 2023

[15] Janaki G, ShanmugaPriya S, "An Analysis on the Ternary Cubic Diophantine Equation "International Journal for Research in Applied Science and Engineering Technology, Volume-11, Issues III, March 2023.



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