Integer Solutions on Ternary Quadratic Diophantine Equation $3(x^2 + y^2) - 5xy = 36z^2$

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Abstract: The non-zero distinct integer solution to the Quadratic Diophantine equation with three unknowns are examined. We obtain the various patterns of integer points that satisfy the ternary Quadratic equation. A few interesting relationship between the solution and the unique pattern.
Keywords: Integral solutions, Ternary Quadratic equation, Polygonal number

I. INTRODUCTION

Number theory is a type of mathematics concerned with the properties and relationship of number, specifically integers. Number theory has a long history and has made significant contributions to many fields of mathematics. A Diophantine equation with three variables is called a Ternary Diophantine equation. Find an integer solution for x, y, z such that the given equation is satisfied in order to solve Ternary Quadratic Diophantine Equation. In mathematics a Diophantine equation is an equation, typically a polynomial equation in two or more unknown with integer coefficients for which only integer solution are of interest. The Diophantine equations provide an infinite fields for study [1-3]. In particular, when solving Quadratic equation involving three unknowns, reference [4-10].This correspondence belongs to an addition exciting equation, which is $3(x^2 + y^2) - 5xy = 36z^2$, which is a homogeneous quadratic equation with three unknown. The Objective is to determine the infinite number of non-zero integral points of this equation. A few amazing relationship between the solution are additionally shown.

II. NOTATION

$T_{m,n} = n\left[1 + \dfrac{(n-1)(m-2)}{2}\right] =$ Polygonl number with rank n and side m.

$O_n = \dfrac{(2n^3 + n)}{3} =$ Octahendral number of rank n.

$4DF_n = \dfrac{n^2(n^2 - 1)}{12} =$ Four dimensional figurate whose generating polygonal is a square.

$SO_n = n(2n^2 - 1) =$ Stella octangula number of rank n.

III. METHOD OF ANALYSIS

To find the non-zero integral solution of the ternary quadratic Diophantine equation

$$3(x^2 + y^2) - 5xy = 36z^2 \qquad (1)$$

The replacement of linear transformations

$$x = p + q \quad \text{and} \quad y = p - q \qquad (2)$$

As (1) result in

$$p^2 + 11q^2 = 36z^2 \qquad (3)$$

A. Pattern 1

consider, $\qquad z = z(a,b) = a^2 + 11b^2$ (4)

When the integer a and b are non-zero

$$36 = (5 + i\sqrt{11})(5 - i\sqrt{11})$$ (5)

By applying factorization techniques and the equation (4) and (5) in (3)

$$(p + i\sqrt{11}q)(p - i\sqrt{11}q) = (5 + i\sqrt{11})(5 - i\sqrt{11})(a + i\sqrt{11}b)^2(a - i\sqrt{11}b)^2$$ (6)

By comparing the real and imaginary components of the equation and equating the like term, we obtain

$$p = 5a^2 - 55b^2 - 22b$$

$$q = a^2 - 11b^2 + 10ab$$

Equation (2) can be solved as an integer by substituting the above values for p and q

$$x = x(a,b) = 6a^2 - 66b^2 - 12ab$$

$$y = y(a,b) = 4a^2 - 44b^2 - 32ab$$

$$z = z(a,b) = a^2 + 11b^2$$

Observation

1. $y(a,a) - 4x(2a,2a) = 216a^2$ Nasty number.

2. $x(1,1) + y(1,1) = -144$ Perfect square.

3. $x(a,a) + y(a,a) + 12T_{26,a} + 66Gno_a \equiv 0 \pmod{66}$

4. $y(a,a) + 8T_{20,a} + 32Gno_a \equiv 0 \pmod{32}$

5. $2x(a,a) + 12T_{24,a} + 60Gno_a \equiv 0 \pmod{60}$

6. $x(2a,1) + y(a,1) - 4T_{16,a} + 16Gno_a \equiv 0 \pmod{126}$

B. Pattern 2

Equation (3) is written as,

$$p^2 - 36z^2 = -11q^2$$ (7)

compute equation (7)

$$(p + 6z)(p - 6z) = q \times -11q$$ (8)

Equation (8) is expressed as a ratio as

$$\frac{(p + 6z)}{-11q} = \frac{q}{(p - 6z)} = \frac{a}{b}$$

The two equations that follow are identical to this

$$bp + 11aq + 6bz = 0$$ (9)

$$ap - bq - 6az = 0$$ (10)

When we apply cross-multiplication, we obtain

$$p = 66a^2 - 6b^2$$

$$q = 11a^2 + b^2$$

$$z = -6ab - 6ab$$

The equivalent integer answer (1) can be obtained by substituting the two variables of p and q which appear previous them

$$x = x(a,a) = 77a^2 - 5a^2$$

$$y = y(a,b) = 55a^2 - 7b^2$$

$$z = z(a,b) = -6ab - 6ab$$

Observation

1. $y(1,1) + z(1,1) = 36$ Perfect square

2. $2x(a,a) + 2y(a,a) + 2z(a,a) = 216$ Cubical integer

3. $2x(a,a) - y(a,a) = 96$ Nasty number

4. $x(a,a) - y(a,a) + z(1,1) - 2T_{26,a} - 11Gno_a \equiv 0 (\bmod 1)$

5. $x(a,a) - z(a,a) - 12T_{16,a} - 36Gno_a \equiv 0 (\bmod 36)$

6. $x(a^2,a) + y(a^2,a) - 3360DF_a - 20T_{14,a} - 50Gno_a \equiv 0 (\bmod 50)$

C. Pattern 3

Equation (3) as, $\qquad p^2 = 36z^2 - 11q^2$

Consider, $\qquad p = p(a,b) = 36a^2 + 11b^2$ (11)

$$(6a + \sqrt{11}b)^2 (6a - \sqrt{11}b)^2 = (6z + \sqrt{11}q)(6z - \sqrt{11}q)$$ (12)

When we compare rational and irrational we get,

$$z = \frac{36a^2 + 11b^2}{6}$$

$$q = 12ab$$

Substituting the above mentioned p and q value into equation (2)

$$x = x(a,b) = 36a^2 - 11b^2 + 12ab$$

$$y = y(a,b) = 36a^2 - 11b^2 - 12ab$$

$$z = z(a,b) = \frac{36a^2 + 11b^2}{6}$$

When we substitute a=6A and b=6B in x, y and z we get

$$x = x(A,B) = 1296A^2 - 396B^2 + 432AB$$

$$y = y(A,B) = 1296A^2 - 396B^2 - 432AB$$

$$z = z(A,B) = 216A^2 + 66B^2$$

Observation

1. $x(a,a) - z(a,a) - 105T_{22,a} - 472Gno_a - a \equiv 0 (\bmod 472)$

2. $x(a,1) + z(a,1) - 108T_{30,a} - 918Gno_a \equiv 0 (\bmod 588)$

3. $x(a,a) + y(a,a) - 66a^2$ Nasty number

4. $x(a,a) + z(1,1) - 222T_{14,a} - 555Gno_a \equiv 0 (\bmod 837)$

5. $z(a,a) - 94T_{8,a} - 94Gno_a \equiv 0 (\bmod 94)$

6. $y(a^2,a) - z(a^2,a) - 5180DF_a + 216SO_a - 412T_{5,a} + 5Gno_a \equiv 0 (\bmod 5)$

D. Pattern 4

Equation (3) can be expressed as follows

$$p^2 + 11q^2 = 36z^2 \times 1$$ (13)

'1' Should be written as

$$1 = \frac{(5 + i\sqrt{11})(5 - i\sqrt{11})}{36}$$ (14)

$$36 = (5 + i\sqrt{11})(5 - i\sqrt{11})$$

we can write using equation(14),(15) and the value of z

$$(p + i\sqrt{11}q)(p - i\sqrt{11}q) = (5 + i\sqrt{11})(5 - i\sqrt{11})(a^2 + \sqrt{11}b^2)^2 \frac{(5 + i\sqrt{11})(5 - i\sqrt{11})}{36}$$

Equating positive factor

$$(p + i\sqrt{11}q) = \frac{1}{6}(5 + i\sqrt{11})(5 + i\sqrt{11})(a + i\sqrt{11}b)^2$$

$$(p + i\sqrt{11}q) = \frac{(14a^2 - 154b^2 - 220ab) + i\sqrt{11}(10a^2 - 110b^2 + 28ab)}{6}$$

When we solve the preceding equation for the real and imaginary part, we obtain

$$p = \frac{(14a^2 - 154b^2 - 220ab)}{6}$$

$$q = \frac{(10a^2 - 110b^2 + 28ab)}{6}$$

Substituting the following values of p and q into equation (2)

$$x = \frac{(24a^2 - 264b^2 - 192ab)}{6}$$

$$y = \frac{(4a^2 - 44b^2 - 248ab)}{6}$$

$$z = a^2 + 11b^2$$

Now if replace a=6A and b=6B in x, y and z we get

$$x = x(A,B) = 144A^2 - 1584B^2 - 1152AB$$

$$y = y(A,B) = 24A^2 - 264B^2 - 1488AB$$

$$z = z(A,B) = 36A^2 - 396B^2$$

Observation

1. $x(a,1) + y(a,1) - 28T_{14,a} + 1250Gno_a \equiv (\mathrm{mod}\, 3098)$

2. $y(a,a) + z(a,a) + 288T_{11,a} + 504Gno_a \equiv 0(\mathrm{mod}\, 504)$

3. $y(a^2,a) - x(a^2,a) + 5760DF_a + 168SO_a - 100T_{26,a} - 466Gno_a \equiv 0(\mathrm{mod}\, 466)$

4. $z(a,a) - 54T_{14,a} - 189Gno_a \equiv 0(\mathrm{mod}\, 189)$

5. $y(a,a) - x(a,a) = 864a^2$ Nasty number

6. $x(a^2,1) - y(a,a) - 6912DF_a + 576SO_a - 64T_{11,a} + 176Gno_a \equiv 0(\mathrm{mod}\, 176)$

IV.   CONCLUSION

This paper presents an unlimited number of different integer solutions, all non-zero, to the ternary quadratic Diophantine problem $3(x^2 + y^2) - 5xy = 36z^2$. Because quadratic equations are various, one can search for other quadratic equation with variable higher than properties using special number.

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