# Integer Solutions on Ternary Quadratic Diophantine Equation $3\left(x^{2}+y^{2}\right)-5 x y=36 z^{2}$ 

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#### Abstract

The non-zero distinct integer solution to the Quadratic Diophantine equation with three unknowns are examined. We obtain the various patterns of integer points that satisfy the ternary Quadratic equation. A few interesting relationship between the solution and the unique pattern.


Keywords: Integral solutions, Ternary Quadratic equation, Polygonal number

## I. INTRODUCTION

Number theory is a type of mathematics concerned with the properties and relationship of number, specifically integers. Number theory has a long history and has made significant contributions to many fields of mathematics. A Diophantine equation with three variables is called a Ternary Diophantine equation. Find an integer solution for $x, y, z$ such that the given equation is satisfied in order to solve Ternary Quadratic Diophantine Equation. In mathematics a Diophantine equation is an equation, typically a polynomial equation in two or more unknown with integer coefficients for which only integer solution are of interest. The Diophantine equations provide an infinite fields for study [1-3]. In particular, when solving Quadratic equation involving three unknowns, reference [4-10].This correspondence belongs to an addition exciting equation, which is $3\left(x^{2}+y^{2}\right)-5 x y=36 z^{2}$, which is a homogeneous quadratic equation with three unknown. The Objective is to determine the infinite number of non-zero integral points of this equation. A few amazing relationship between the solution are additionally shown.

## II. NOTATION

$T_{m, n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]=$ Polygonl number with rank n and side m .
$O_{n}=\frac{\left(2 n^{3}+n\right)}{3}=$ Octahendral number of rank n .
$4 D F_{n}=\frac{n^{2}\left(n^{2}-1\right)}{12}=$ Four dimensional figurate whose generating polygonal is a square.
$S O_{n}=n\left(2 n^{2}-1\right)=$ Stella octangula number of rank $n$.

## III. METHOD OF ANALYSIS

To find the non-zero integral solution of the ternary quadratic Diophantine equation

$$
\begin{equation*}
3\left(x^{2}+y^{2}\right)-5 x y=36 z^{2} \tag{1}
\end{equation*}
$$

The replacement of linear transformations
$x=p+q$ and $y=p-q$
As (1) result in

$$
\begin{equation*}
p^{2}+11 q^{2}=36 z^{2} \tag{3}
\end{equation*}
$$

A. Pattern 1

$$
\begin{equation*}
\text { consider, } \quad z=z(\mathrm{a}, b)=\mathrm{a}^{2}+11 b^{2} \tag{4}
\end{equation*}
$$

When the integer $a$ and $b$ are non-zero

$$
\begin{equation*}
36=(5+i \sqrt{11})(5-i \sqrt{11}) \tag{5}
\end{equation*}
$$

By applying factorization techniques and the equation (4) and (5) in (3)

$$
\begin{equation*}
(p+i \sqrt{11} q)(p-i \sqrt{11} q)=(5+i \sqrt{11})(5-i \sqrt{11})(\mathrm{a}+\mathrm{i} \sqrt{11} b)^{2}(\mathrm{a}-\mathrm{i} \sqrt{11} b)^{2} \tag{6}
\end{equation*}
$$

By comparing the real and imaginary components of the equation and equating the like term, we obtain

$$
\begin{aligned}
& p=5 \mathrm{a}^{2}-55 b^{2}-22 b \\
& q=\mathrm{a}^{2}-11 b^{2}+10 a b
\end{aligned}
$$

Equation (2) can be solved as an integer by substituting the above values for p and q

$$
\begin{aligned}
& x=x(\mathrm{a}, b)=6 \mathrm{a}^{2}-66 b^{2}-12 \mathrm{ab} \\
& y=y(\mathrm{a}, \mathrm{~b})=4 \mathrm{a}^{2}-44 b^{2}-32 \mathrm{ab} \\
& z=z(\mathrm{a}, \mathrm{~b})=\mathrm{a}^{2}+11 b^{2}
\end{aligned}
$$

## Observation

1. $y(\mathrm{a}, \mathrm{a})-4 \mathrm{x}(2 \mathrm{a}, 2 \mathrm{a})=216 \mathrm{a}^{2}$ Nasty number.
2. $x(1,1)+y(1,1)=-144$ Perfect square.
3. $x(\mathrm{a}, \mathrm{a})+\mathrm{y}(\mathrm{a}, \mathrm{a})+12 \mathrm{~T}_{26, \mathrm{a}}+66$ Gno $_{\mathrm{a}} \equiv 0(\bmod 66)$
4. $y(\mathrm{a}, \mathrm{a})+8 \mathrm{~T}_{20, \mathrm{a}}+32 G n o_{\mathrm{a}} \equiv 0(\bmod 32)$
$5.2 x(\mathrm{a}, \mathrm{a})+12 \mathrm{~T}_{24, \mathrm{a}}+60 G n o_{a} \equiv 0(\bmod 60)$
5. $x(2 \mathrm{a}, 1)+\mathrm{y}(\mathrm{a}, 1)-4 \mathrm{~T}_{16, \mathrm{a}}+16$ Gno $_{\mathrm{a}} \equiv 0(\bmod 126)$

## B. Pattern 2

Equation (3) is written as,

$$
\begin{equation*}
p^{2}-36 z^{2}=-11 q^{2} \tag{7}
\end{equation*}
$$

compute equation (7)

$$
\begin{equation*}
(p+6 z)(p-6 z)=q \times-11 q \tag{8}
\end{equation*}
$$

Equation (8) is expressed as a ratio as

$$
\frac{(\mathrm{p}+6 \mathrm{z})}{-11 \mathrm{q}}=\frac{q}{(p-6 z)}=\frac{\mathrm{a}}{\mathrm{~b}}
$$

The two equations that follow are identical to this

$$
\begin{align*}
& b p+11 \mathrm{aq}+6 \mathrm{bz}=0  \tag{9}\\
& \mathrm{ap}-\mathrm{bq}-6 \mathrm{az}=0 \tag{10}
\end{align*}
$$

When we apply cross-multiplication, we obtain

$$
\begin{aligned}
& p=66 a^{2}-6 \mathrm{~b}^{2} \\
& q=11 \mathrm{a}^{2}+b^{2} \\
& z=-6 \mathrm{a} b-6 \mathrm{ab}
\end{aligned}
$$

The equivalent integer answer (1) can be obtained by substituting the two variables of p and q which appear previous them

$$
\begin{aligned}
& x=x(\mathrm{a}, \mathrm{a})=77 \mathrm{a}^{2}-5 \mathrm{a}^{2} \\
& y=y(\mathrm{a}, \mathrm{~b})=55 \mathrm{a}^{2}-7 \mathrm{~b}^{2} \\
& z=z(\mathrm{a}, \mathrm{~b})=-6 \mathrm{ab}-6 \mathrm{ab}
\end{aligned}
$$

Observation

1. $y(1,1)+z(1,1)=36$ Perfect square
2. $2 x(\mathrm{a}, \mathrm{a})+2 \mathrm{y}(\mathrm{a}, \mathrm{a})+2 \mathrm{z}(\mathrm{a}, \mathrm{a})=216$ Cubical integer
3. $2 x(\mathrm{a}, \mathrm{a})-\mathrm{y}(\mathrm{a}, \mathrm{a})=96$ Nasty number
4. $x(\mathrm{a}, \mathrm{a})-\mathrm{y}(\mathrm{a}, \mathrm{a})+\mathrm{z}(1,1)-2 \mathrm{~T}_{26, \mathrm{a}}-11 G n o_{\mathrm{a}} \equiv 0(\bmod 1)$
5. $x(\mathrm{a}, \mathrm{a})-\mathrm{z}(\mathrm{a}, \mathrm{a})-12 \mathrm{~T}_{16, \mathrm{a}}-36 G n o_{\mathrm{a}} \equiv 0(\bmod 36)$
6. $x\left(\mathrm{a}^{2}, \mathrm{a}\right)+\mathrm{y}\left(\mathrm{a}^{2}, \mathrm{a}\right)-3360 \mathrm{DF}_{\mathrm{a}}-20 T_{14, \mathrm{a}}-50 G n o_{\mathrm{a}} \equiv 0(\bmod 50)$
C. Pattern 3

Equation (3) as,

$$
p^{2}=36 z^{2}-11 q^{2}
$$

Consider,

$$
\begin{gather*}
p=p(\mathrm{a}, \mathrm{~b})=36 \mathrm{a}^{2}+11 b^{2}  \tag{11}\\
(6 \mathrm{a}+\sqrt{11} b)^{2}(6 \mathrm{a}-\sqrt{11} b)^{2}=(6 z+\sqrt{11} q)(6 z-\sqrt{11} q) \tag{12}
\end{gather*}
$$

When we compare rational and irrational we get,

$$
\begin{aligned}
& z=\frac{36 \mathrm{a}^{2}+11 b^{2}}{6} \\
& q=12 \mathrm{ab}
\end{aligned}
$$

Substituting the above mentioned $p$ and $q$ value into equation (2)

$$
\begin{aligned}
& x=x(\mathrm{a}, \mathrm{~b})=36 \mathrm{a}^{2}-11 b^{2}+12 \mathrm{ab} \\
& y=y(\mathrm{a}, \mathrm{~b})=36 \mathrm{a}^{2}-11 b^{2}-12 \mathrm{ab} \\
& z=z(\mathrm{a}, \mathrm{~b})=\frac{36 \mathrm{a}^{2}+11 b^{2}}{6}
\end{aligned}
$$

When we substitute $a=6 A$ and $b=6 B$ in $x, y$ and $z$ we get

$$
\begin{aligned}
& x=x(\mathrm{~A}, \mathrm{~B})=1296 \mathrm{~A}^{2}-396 B^{2}+432 \mathrm{AB} \\
& y=y(\mathrm{~A}, \mathrm{~B})=1296 \mathrm{~A}^{2}-396 B^{2}-432 \mathrm{AB} \\
& z=z(\mathrm{~A}, \mathrm{~B})=216 \mathrm{~A}^{2}+66 B^{2}
\end{aligned}
$$

## Observation

1. $x(\mathrm{a}, \mathrm{a})-\mathrm{z}(\mathrm{a}, \mathrm{a})-105 \mathrm{~T}_{22, \mathrm{a}}-472 G n o_{a}-a \equiv 0(\bmod 472)$
2. $x(\mathrm{a}, 1)+\mathrm{z}(\mathrm{a}, 1)-108 \mathrm{~T}_{30, \mathrm{a}}-918 G n o_{\mathrm{a}} \equiv 0(\bmod 588)$
3. $x(a, a)+y(a, a)-66 a^{2}$ Nasty number
4. $x(\mathrm{a}, \mathrm{a})+\mathrm{z}(1,1)-222 \mathrm{~T}_{14, \mathrm{a}}-555$ Gno $_{\mathrm{a}} \equiv 0(\bmod 837)$
5. $z(\mathrm{a}, \mathrm{a})-94 \mathrm{~T}_{8, \mathrm{a}}-94 G n o_{\mathrm{a}} \equiv 0(\bmod 94)$
6. $y\left(\mathrm{a}^{2}, \mathrm{a}\right)-\mathrm{z}\left(\mathrm{a}^{2}, \mathrm{a}\right)-5180 \mathrm{DF}_{\mathrm{a}}+216 S O_{\mathrm{a}}-412 T_{5, \mathrm{a}}+5 G n o_{\mathrm{a}} \equiv 0(\bmod 5)$

## D. Pattern 4

Equation (3) can be expressed as follows

$$
\begin{equation*}
p^{2}+11 q^{2}=36 z^{2} \times 1 \tag{13}
\end{equation*}
$$

' 1 ' Should be written as

$$
\begin{equation*}
1=\frac{(5+i \sqrt{11})(5-i \sqrt{11})}{36} \tag{14}
\end{equation*}
$$

$$
36=(5+i \sqrt{11})(5-i \sqrt{11})
$$

we can write using equation(14),(15) and the value of $z$

$$
(p+i \sqrt{11} q)(p-i \sqrt{11} q)=(5+i \sqrt{11})(5-i \sqrt{11})\left(\mathrm{a}^{2}+\sqrt{11} b^{2}\right)^{2} \frac{(5+i \sqrt{11})(5-i \sqrt{11})}{36}
$$

Equating positive factor

$$
\begin{gathered}
(p+i \sqrt{11} q)=\frac{1}{6}(5+i \sqrt{11})(5+i \sqrt{11})(\mathrm{a}+\mathrm{i} \sqrt{11} b)^{2} \\
(p+i \sqrt{11} q)=\frac{\left(14 \mathrm{a}^{2}-154 b^{2}-220 \mathrm{ab}\right)+\mathrm{i} \sqrt{11}\left(10 \mathrm{a}^{2}-110 b^{2}+28 \mathrm{ab}\right)}{6}
\end{gathered}
$$

When we solve the preceding equation for the real and imaginary part, we obtain

$$
\begin{aligned}
& p=\frac{\left(14 \mathrm{a}^{2}-154 b^{2}-220 \mathrm{ab}\right)}{6} \\
& q=\frac{\left(10 \mathrm{a}^{2}-110 b^{2}+28 \mathrm{ab}\right)}{6}
\end{aligned}
$$

Substituting the following values of p and q into equation (2)

$$
\begin{aligned}
x & =\frac{\left(24 \mathrm{a}^{2}-264 b^{2}-192 \mathrm{ab}\right)}{6} \\
y & =\frac{\left(4 \mathrm{a}^{2}-44 b^{2}-248 \mathrm{ab}\right)}{6} \\
& z=\mathrm{a}^{2}+11 b^{2}
\end{aligned}
$$

Now if replace $a=6 A$ and $b=6 B$ in $x, y$ and $z$ we get

$$
\begin{aligned}
& x=x(\mathrm{~A}, \mathrm{~B})=144 \mathrm{~A}^{2}-1584 \mathrm{~B}^{2}-1152 \mathrm{AB} \\
& y=y(\mathrm{~A}, \mathrm{~B})=24 \mathrm{~A}^{2}-264 B^{2}-1488 \mathrm{AB} \\
& z=z(\mathrm{~A}, \mathrm{~B})=36 \mathrm{~A}^{2}-396 B^{2}
\end{aligned}
$$

## Observation

1. $x(\mathrm{a}, 1)+\mathrm{y}(\mathrm{a}, 1)-28 \mathrm{~T}_{14, \mathrm{a}}+1250 G n o_{\mathrm{a}} \equiv(\bmod 3098)$
2. $y(\mathrm{a}, \mathrm{a})+\mathrm{z}(\mathrm{a}, \mathrm{a})+288 \mathrm{~T}_{11, \mathrm{a}}+504 G n o_{\mathrm{a}} \equiv 0(\bmod 504)$
3. $y\left(\mathrm{a}^{2}, \mathrm{a}\right)-\mathrm{x}\left(\mathrm{a}^{2}, \mathrm{a}\right)+5760 \mathrm{DF}_{\mathrm{a}}+168 S O_{\mathrm{a}}-100 T_{26, \mathrm{a}}-466$ Gno $_{\mathrm{a}} \equiv 0(\bmod 466)$
4. $z(\mathrm{a}, \mathrm{a})-54 \mathrm{~T}_{14, \mathrm{a}}-189 G n o_{\mathrm{a}} \equiv 0(\bmod 189)$
5. $y(a, a)-x(a, a)=864 a^{2}$ Nasty number
6. $x\left(\mathrm{a}^{2}, 1\right)-y(\mathrm{a}, \mathrm{a})-6912 \mathrm{DF}_{\mathrm{a}}+576 \mathrm{SO}_{\mathrm{a}}-64 T_{11, \mathrm{a}}+176 G n o_{\mathrm{a}} \equiv 0(\bmod 176)$

## IV. CONCLUSION

This paper presents an unlimited number of different integer solutions, all non-zero, to the ternary quadratic Diophantine problem $3\left(x^{2}+y^{2}\right)-5 x y=36 z^{2}$. Because quadratic equations are various, one can search for other quadratic equation with variable higher than properties using special number.

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