# Integral Solutions of the Bi-Quadratic Equation 

$$
2\left(x^{2}-y^{2}\right)\left(7 x^{2}+y^{2}\right)=520\left(z^{2}-w^{2}\right) t^{2}+24 s^{2}
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Abstract: The biquadratic with six unknowns of the form $2\left(x^{2}-y^{2}\right)\left(7 x^{2}+y^{2}\right)=520\left(z^{2}-w^{2}\right) t^{2}+24 s^{2}$ has been studied for its non-trivial distinct integral solutions. A few interesting relations among the solutions and polygonal numbers are presented.

## I. INTRODUCTION

Carl Friedrich Gauss, a well-known mathematician who lived from 1777 to 1855 , is credited with saying that "number theory is the queen of mathematics and mathematics is the queen of all sciences." The study of integer and rational number features that go beyond standard manipulations in common arithmetic is known as number theory, or higher arithmetic. Number theorists were created in practically every major civilization throughout history. In the ancient and medieval eras, these individuals were typically geometers, but they may also have been philosophers, calendar calculators, astronomers, astrologers, priests,or magicians. Due to their variety, the quadratic offers an infinite topic for study. One may specifically refer to [4] through [20]. This problem of communication with the non-homogeneous quadratic equation denoted by,

$$
2\left(x^{2}-y^{2}\right)\left(7 x^{2}+y^{2}\right)=520\left(z^{2}-w^{2}\right) t^{2}+24 s^{2}
$$

is examined for its distinct non-zero integral solution. Some intriguing connections between the solutions are presented.

## II. NOTATIONS

| SPECIAL NUMBER | NOTATIONS | DEFINITIONS |
| :--- | :--- | :--- |
| Triangular number | $T_{3, n}$ | $\frac{n(n+1)}{2}$ |
| Hexadecagonal number | $T_{16, n}$ | $n(7 n-6)$ |
| Nonadecagonal number | $T_{19, n}$ | $\frac{n(17 n-15)}{2}$ |
| Icositetragonal number | $T_{24, n}$ | $n(11 n-10)$ |
| Octahedral number | $O_{n}$ | $\frac{1}{3}\left(2 n^{3}+n\right)$ |
| Centered square number | $C S_{n}$ | $n^{2}+(n-1)^{2}$ |
| Gnomonic number | $G n o_{n}$ | $(2 n-1)$ |
| Stella octangular number | $S O_{n}$ | $n\left(2 n^{2}-1\right)$ |
| Star number | $S t a r_{n}$ | $6 n(n-1)+1$ |


| Rhombic dodecagonal number | $R D_{n}$ | $(2 n-1)\left(2 n^{2}-2 n+1\right)$ |
| :--- | :--- | :--- |
| Pentadecagonal number | $T_{15, n}$ | $\frac{n(13 n-11)}{2}$ |

## III. METHOD OF ANALYSIS

The Quadratic equation under consideration is
$2\left(x^{2}-y^{2}\right)\left(7 x^{2}+y^{2}\right)=520\left(z^{2}-w^{2}\right) t^{2}+24 s^{2}$
On substituting the linear transformation,
$x=u+v ; y=u-v$;
$z=2 u+v ; w=2 u-v ; s=2 u v$
in (1), leads to
$u^{2}+v^{2}=65^{2}$
Equation (3) is solved through different methods and thus we obtain four different patterns of integral solutions to (1)

## A. Pattern I

Assume
$t=t(a, b)=a^{2}+b^{2}$
Where a and b are non-zero integers.
Write $65=(1+8 i)(1-8 i)$
(5) Applying the
factorization method and substituting (4) and (5) in (3),
$(u+i v)(u-i v)=(1+8 i)(1-8 i)(a+i b)^{2}(a-i b)^{2}$
Equating real and imaginary parts, the values of $u$ and $v$ are
$u=a^{2}-b^{2}-16 a b$
$v=8 a^{2}-8 b^{2}+2 a b$
Substituting the values of u and v in equation (2), the non zero distinct integral solutions of (1) are executed as,
$x=x(a, b)=9 a^{2}-9 b^{2}-14 a b$
$y=y(a, b)=-7 a^{2}+7 b^{2}-18 a b$
$z=z(a, b)=10 a^{2}-10 b^{2}-30 a b$
$w=w(a, b)=-6 a^{2}+6 b^{2}-34 a b$
$s=s(a, b)=16 a^{4}-96 a^{2} b^{2}-252 a^{3} b+252 a b^{3}+16 b^{4}$
$t=t(a, b)=a^{2}+b^{2}$
Thus (7) gives the integral solution of (1)

## Observations

1. $2 s(1,1)+2 t(1,1)-x(1,1)$ is a perfect square.
2. $5 x(1,-1)+2 z(1,-1)+2 w(1,-1)+t(2,1)$ is a happy couple number
3. $z\left(A^{2}, 4\right)-480 D F_{A}+10 T_{24, A}+50 G n o_{A} \equiv 0(\bmod 210)$
4. $8 x(2 A, A)-3 z(A, A)+6 t(2 A, A)+w(A, A)-12 T_{15, A}-33 G n o_{A} \equiv 0(\bmod 33)$
5. $2 s(A, 1)-5 y(2 A+1, A)+t(A, A)-1536 D F_{A}+756 O_{A}-43 T_{16, A}-622 D G n o_{A} \equiv 0(\bmod 689)$
6. $4 t(B+1, B)+s(B, 2)-768 D F_{B}+756 O_{B}+4 T_{8, B}-1134 G n o_{B} \equiv 0(\bmod 1394)$
7. $2 w(3 A, 1)+3 z(A+2, A)-162 \mathrm{SO}_{4}-5$ Star $_{A}+66$ Gno $_{A} \equiv 0(\bmod 443)$

## B. Pattern II

Assume
$t=t(a, b)=a^{2}+b^{2}$
(8) Where a and b are
non-zero integers.
Write $65=(8+i)(8-i)$
(9) Applying
the factorization method and substituting (8) and (9) in (3),
$(u+i v)(u-i v)=(8+i)(8-i)(a+i b)^{2}(a-i b)^{2}$
(10) Equating real and
imaginary parts, the values of $u$ and $v$ are
$u=8 a^{2}-8 b^{2}-2 a b$
$v=a^{2}-b^{2}+16 a b$
Substituting the values of $u$ and $v$ in equation (2), the non zero distinct integral solutions of (1) are executed as,
$x=x(a, b)=9 a^{2}-9 b^{2}+14 a b$
$y=y(a, b)=7 a^{2}-7 b^{2}-18 a b$
$z=z(a, b)=17 a^{2}-17 b^{2}-12 a b$
$w=w(a, b)=15 a^{2}-15 b^{2}-20 a b$
$s=s(a, b)=16 a^{4}-96 a^{2} b^{2}+252 a^{3} b-252 a b^{3}+16 b^{4}$
$t=t(a, b)=a^{2}+b^{2}$
gives the distinct integral solution of (1)

## Observations

1. $4 x(1,1)-4 y(1,1)+z(1,1)+2 t(1,1)$ is a perfect square.
2. $4 x(1,1)-2 y(1,1)+12 z(1,1)-4 w(1,1)-4 s(1,1)-30 t(1,1)$ is a cubical number.
3. $s(1,1)+z(1,1)+t(5,0)$ is a palindrom number.
4. $y(2 A+1,2)-4 T_{16, A}+10$ Gno $_{A} \equiv 0(\bmod 67)$
5. $w(A+1, A)+z(2 A-1, A)-15$ Star $_{A}+20$ Gno $_{A} \equiv 0(\bmod 323)$
6. $10 w(2 A, A)+t\left(2 A^{2}, 3\right)+z(2 A, 1)-192 D F_{A}-68 T_{19, A}-G n o_{A} \equiv 0(\bmod 248)$
7. $y\left(2 A-1, A^{2}\right)+10 t\left(A^{2}, A\right)+z(A, 0)-144 D F_{A}+54 O_{A}-152 T_{3, A}+43 G n o_{A} \equiv 0(\bmod 36)$
C. Pattern III

Assume
$t=t(a, b)=a^{2}+b^{2}$
Where a and b are non-zero integers.
Write $\quad 65=(4+7 i)(4-7 i)$
(13) Applying the
factorization method and substituting (12) and (13) in (3),
$(u+i v)(u-i v)=(4+7 i)(4-7 i)(a+i b)^{2}(a-i b)^{2}$
Equating real and imaginary parts, the values of u and v are
$u=4 a^{2}-4 b^{2}-14 a b$
$v=7 a^{2}-7 b^{2}+8 a b$
Substituting the values of u and v in equation (2), the non zero distinct integral solutions of (1) are executed as,
$x=x(a, b)=11 a^{2}-11 b^{2}-6 a b$
$y=y(a, b)=-3 a^{2}+3 b^{2}-22 a b$
$z=z(a, b)=15 a^{2}-15 b^{2}-20 a b$
$w=w(a, b)=a^{2}-b^{2}-36 a b$
$s=s(a, b)=56 a^{4}-336 a^{2} b^{2}-132 a^{3} b+132 a b^{3}+56 b^{4}$
$t=t(a, b)=a^{2}+b^{2}$
the distinct integral solution of (1)

## Observations

1. $\frac{1}{2} x(1,-1)-y(1,1)+z(-1,1)+w(1,-1)$ is a perfect square.
2. $z(1,-1)-2 s(1,1)+4 t(1,1)$ is a cubical number.
3. $x(1,1)+10 t(5,4)$ is a Duck number.
4. $z\left(2 A+1, A^{2}\right)+720 D F_{A}+60 O_{A}-5 T_{12, A}-50 G n o_{A} \equiv 0(\bmod 65)$
5. $x\left(2 A-1, A^{2}\right)+2 y(A+1,2)+528 D F_{A}+3 R D_{A}+3 T_{12, A}+60 G n o_{A} \equiv 0(\bmod 122)$
6. $x(2 A-1, A)+2 y(A+1,2)+528 D F_{A}+3 R D_{A}-3 T_{12, A}+60 G n o_{A} \equiv 0(\bmod 122)$
7. $s(1, A)-2688 D F_{A}-66 S O_{A}+40 T_{16, A}+153 G n o_{A} \equiv 0(\bmod 97)$
D. Pattern 4
(3) can be written as
$u^{2}=65 t^{2}-v^{2}$
(16) Assuming $\quad u=65 a^{2}-b^{2}$
and

$$
\begin{equation*}
1=\left(\frac{(\sqrt{65}+1)(\sqrt{65}-1)}{64}\right) \tag{17}
\end{equation*}
$$

(17) Substituting (16) in
using the method of factorization

$$
\frac{(\sqrt{65}+1)(\sqrt{65}-1)(\sqrt{65} a-b)^{2}(\sqrt{65} a+b)^{2}}{64}=(\sqrt{65} t+v)(\sqrt{65} t-v)
$$

Now define

$$
\begin{align*}
& (\sqrt{65} t+v)=\frac{(\sqrt{65}+1)(\sqrt{65} a+b)^{2}}{8} \\
& (\sqrt{65} t-v)=\frac{(\sqrt{65}-1)(\sqrt{65} a-b)^{2}}{8} \tag{18}
\end{align*}
$$

Equating the like in (18) we get
$t=\frac{1}{8}\left(65 a^{2}+b^{2}+2 a b\right)$
$v=\frac{1}{8}\left(65 a^{2}+b^{2}+130 a b\right)$
Since our interest is on finding integer solution we have choose a and b Suitably so that t and v integers,
$t=65 A^{2}+B^{2}+2 A B$
$v=65 A^{2}+B^{2}+130 A B$
$u=520 A^{2}-8 B^{2}$
Thus, using the values of u and v performing a few calculations the
Values of $x, y$ and $z$ are obtained as follows:
$x=x(A, B)=585 A^{2}-7 B^{2}+130 A B$
$y=y(A, B)=455 A^{2}-9 B^{2}-130 A B$
$z=z(A, B)=1105 A^{2}-15 B^{2}+130 A B$
$w=w(A, B)=975 A^{2}-17 B^{2}-130 A B$
$s=s(A, B)=67600 A^{2}+135200 A^{3} B-2080 A B^{3}-16 B^{4}$
$t=t(A, B)=65 A^{2}+B^{2}+2 A B$
Thus (19) represents the non-trivial integral solution of (1)

## Observations

1. $w(1,1)-y(1,1)$ is a cubical number.
2. $x(1,1)-t(1,1)-y(1,1)$ is a perfect square.
3. $s(1,1)+t(1,1)$ is a Duck number.
4. $x(A, 1)+t(A, A)+y(1, A)-116 T_{13, A}-391 G n o_{A} \equiv 0(\bmod 1489)$
5. $w(A, 2)+t(2 A, 1)-y(0, A)-622 C S_{A}-183 G n o_{A} \equiv 0(\bmod 506)$
6. $x(1, A)-t(2,2 A)+T_{24, A}-56 G n o_{A} \equiv 0(\bmod 381)$
7. $y(2 A, 1)-728 T_{7, A}-416 G n o_{A} \equiv 0(\bmod 407)$

## IV. CONCLUSION

In this study, an attempt has been made to complete the set of non-trivial distinct integral solutions for the non-homogeneous biquadratic equation. Finally, in order to find more solution to the biquadratic equation under consideration, one can look for further biquadratic equations that involve many variables.

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