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Intrinsic Solution on Quadratic Diophantine Equation $z^2 = 170x^2 + y^2$

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***Abstract:** The non-zero unique integer solutions of the Quadratic Diophantine Equation with three unknowns $z^2 = 170x^2 + y^2$ are examined. There are several absorbing relationships between the answers and a few unique numbers of rank n that arespecial Polygonal numbers, Star number and Gnomonic numbers.*

***Keywords:** Three-variable quadratic equation, integral solutions, Polygonal numbers, Star number and Gnomonic number.*

I. INTRODUCTION

There are various distinct kinds of quadratic Diophantine equations. Anyone can look at [1,7] for an in-depth review of a broad spectrum of concerns and a substantial survey of the body of literature. The non-trivial integral solutions to the quadratic Diophantine problem of the form $lxy + m(x, y) = z^2$ have been examined in [8]. The integral solutions of an distinct Pythagorean triangular problem have been investigated in [9,10]. Two parametric non-trivial integral solutions to the quadratic homogeneity Diophantine problem $X^2 + PXY + Y^2 = Z^2$, for which P constitutes a non-zero constant, are given in [11]. The non-trivial integral solutions of the quadratic homogeneity equation, $l\alpha(x^2 + y^2) + bxy = 4l\alpha^2 z^2$ have been analysed in [12]. This formof quadratic Diophantine equation, $(x - y)(x - z) + y^2 = 0$ is examined in [13] for its integral solutions at various angles, and their parameterized representations are identified.

we examine an additional fascinating quadratic equation, $z^2 = 170x^2 + y^2$, and derive distinct patterns of integral solutions that are non-trivial. Furthermore, a few fascinating relationships between the solutions special Polygonal number and Gnomonic numbers are illustrated.

II. NOTATIONS

$T_{m,n} = \left[1 + \frac{(n-1)(m-2)}{2}\right]$ = Polygonal Number with rank n and sides m.

$Star_n = 6n(n-1) + 1$ = Star number of rank n.

$Gno_n = (2n-1)$ = Gnomonic number of rank n.

III. METHOD OF ANALYSIS

The Diophantine equation of quadratic has to be computed for its non-zero integral solution is

$$z^2 = 170x^2 + y^2 \quad (1)$$

Assuming,

$$z = z(a,b) = a^2 + 170b^2 \quad (2)$$

where a and b are non-zero integers.

A. Pattern: 1

Equation (1) can be written as

$$z^2 - 170x^2 = y^2 \quad (3)$$

Assuming $y = y(a,b) = a^2 - 170b^2$ (4)

We get,

$$\left(a^2 + 170b^2\right)^2 = \left(170x^2 + y^2\right)$$

Using the factorization method, we have

$$z + \sqrt{170}x = \left(a + \sqrt{170}b\right)^2 \quad (5)$$

$$z - \sqrt{170}x = \left(a - \sqrt{170}b\right)^2 \quad (6)$$

Comparing the rational and irrational factors,

$$z = z(a,b) = a^2 + 170b^2$$

$$x = x(a,b) = 2ab$$

The corresponding non-zero distinct solutions are

$$x = x(a,b) = 2ab$$

$$y = y(a,b) = a^2 - 170b^2$$

$$z = z(a,b) = a^2 + 170b^2$$

Observations:

1. If $a = b$ and a is even, then a is divisible by 2.
2. If a is odd and b is even, then z is divisible by 3.
3. $x(b,b) + y(b,b) + z(b,b)T_{10,b} \equiv 0 (\bmod 3)$
4. $x(1,a) - y(1,a) + z(1,a) - 68T_{12,a} - 137GnO_a \equiv 0 (\bmod 137)$
5. $y(1,1) + z(1,1) + x(1,1)$ is a perfect square.
6. $y(2,2) + z(2,2)$ is a Deficient number.
7. $11x(b,b) + y(b,b) + z(b,b)$ which represents a Nasty number.
8. For any values of a and b, $z - x - y$ is divisible by 2.

B. Pattern: 2

Equation (1) can be written as

$$170x^2 + y^2 = z^2 * 1 \quad (7)$$

Assuming

$$z = z(a,b) = a^2 + 170b^2$$

and write

$$1 = \frac{\left(13 + i12\sqrt{170}\right)\left(13 - i12\sqrt{170}\right)}{24649} \quad (8)$$

Using factorization method, equation (7) can be written as

$$\left(y + i\sqrt{170}x\right)\left(y - i\sqrt{170}x\right) = \left[\frac{13 + i12\sqrt{170}}{157}\right]\left[\frac{13 - i12\sqrt{170}}{157}\right]\left[\left(a + i\sqrt{170}b\right)^2\left(a - i\sqrt{170}b\right)^2\right]$$

we get

$$\left(y + i\sqrt{170}x\right) = \frac{1}{157}\left(13 + i12\sqrt{170}\right)\left(a - i\sqrt{170}b\right)^2 \quad (9)$$

$$\left(y-i\sqrt{170}x\right)=\frac{1}{157}\left(13+i12\sqrt{170}\right)\left(a+i\sqrt{170}b\right)^2 \quad (10)$$

Comparing real and imaginary parts, we get

$$x=\frac{1}{157}\left[12a^2-2040b^2+26ab\right]$$

$$y=\frac{1}{157}\left[13a^2-2210b^2-4080ab\right]$$

Since finding integer solutions is what we specialize in, we have chosen and appropriately such that and are integers.

Consider $a=157A$ and $b=157B$

The integer solutions are

$$x=1884A^2-320280B^2+4082AB$$

$$y=2041A^2-346970B^2-640560AB$$

$$z=24649A^2+4190330B^2$$

Observations

1. If A=B, then x-y-z is divisible by 2.
2. $x(A,A)-y(A,A)-134235T_{12,A}-268470GnO_A\equiv 0(\mathrm{mod}\,268470)$
3. $z(B,B)-y(B,B)-x(B,B)-393913T_{30,A}\equiv 0(\mathrm{mod}5120869)$
4. $z(A,1)+x(A,1)-2041T_{28,A}-14287GnO_A\equiv 0(\mathrm{mod}3884337)$
5. $y(B,B)+z(B,B)-293590T_{24,A}\equiv 0(\mathrm{mod}2935900)$

C. Pattern: 3

Equation (1) can be written as

$$z^2-y^2=170x^2$$

and we get

$$(z+y)(z-y)=170x\cdot x \quad (11)$$

a) Case 1

Equation (1) can be written as

$$\frac{z+y}{170x}=\frac{x}{z-y}=\frac{P}{Q} \quad (12)$$

From equation (12), we get two equations

$$-170Px+Qy+Qz=0$$

$$Qx+Py-Pz=0$$

Applying cross ratio method, we get the integer solutions are

$$x=x(P,Q)=-2PQ$$

$$y=y(P,Q)=Q^2-170P^2$$

$$z=z(P,Q)=-Q^2-170P^2$$

Observations

1.For all values of P and Q, x + y – z is divisible by 2.

2. $2y(1,P)-x(1,P)-2z(1,P)-2T_{6,P}-2GnO_P \equiv 0(\text{mod}\,2)$

3. $x(P,1)-4y(P,1)-4z(P,1)-170T_{18,P}-594GnO_P \equiv 0(\text{mod}\,594)$

4. Each of the following expressions

(i) $z(Q,Q)+2x(Q,Q)-y(Q,Q)$

(ii) $T_{30,P}-3x(Q,1)+T_{18,P}$

represents a nasty number.

b) Case 2

Equation (11) can be written as

$$\frac{z+y}{x}=\frac{170x}{z-y}=\frac{P}{Q} \qquad (13)$$

From equation (13), we have two equations

$$-Px+Qy+Qz=0$$

$$170Qx+Py-Pz=0$$

Applying cross ratio method, we get the integer solutions are

$$x=x(P,Q)=-2QP$$

$$y=y(P,Q)=170Q^2-P^2$$

$$z=z(P,Q)=-P^2-170Q^2$$

Observations

1. $y(1,Q)-x(1,Q)-9z(1,Q)-680T_{7,Q}-511GnO_Q \equiv 0(\text{mod } 519)$
2. $10x(1,Q)+3y(1,Q)-85\,Star_Q-245GnO_Q \equiv 0(\text{mod } 257)$
3. $y(Q,Q)-6x(Q,Q)-z(Q,Q)-44T_{18,Q}-154GnO_Q \equiv 0(\text{mod } 154)$
4. $3T_{16,P}-3x(P,1)$ represents a Nasty number.
5. $y(1,1)-x(1,1)$ represents a Palindromic number.
6. $z(1,1)+y(1,1)-5x(1,1)$ is a Dudeney number.

D. Pattern: 4

Equation (1) can be written as

$$z^2-170x^2=y^2*1 \qquad (14)$$

and write

$$1=(\sqrt{170}+13)(\sqrt{170}-13)$$

Assuming,

$$y=y(a,b)=a^2-170b^2 \quad (15)$$

We have

$$z^2-170x^2=(a^2-170b^2)\,[(\sqrt{170}+13)(\sqrt{170}-13)]$$

Using factorization method, we get

$$z+\sqrt{170}x=(a+\sqrt{170}b)^2\,[(\sqrt{170}+13)] \quad (16)$$

$$z-\sqrt{170}x=\left(a+\sqrt{170}b\right)^2\left[\left(\sqrt{170}+13\right)\right] \qquad (17)$$

From equation (16), we get

Thus, the corresponding non-zero distinct integer solutions are

$$x=x(a,b)=a^2+170b^2+26ab$$

$$y=y(a,b)=a^2-170b^2$$

$$z=z(a,b)=13a^2+2210b^2+340ab$$

Observations

1. $z(b,b)-x(b,b)-169T_{30,b}\equiv 0(\text{mod } 2197)$
2. $z(a,1)-y(a,1)+x(a,1)-T_{28,a}-189GnO_a\equiv 0(\text{mod } 2739)$
3. $z(b,b)-y(b,b)-x(b,b)-507T_{12,b}-1014GnO_b\equiv 0(\text{mod } 1014)$
4. $x(1,1)+y(1,1)$ represents a perfect number.
5. $x(1,1)-y(1,1)$ representsa even composite number.

IV. CONCLUSION

The Diophantine quadratic equation using special polygonal numbers has been presented. Futhermore, one can look for integer solutions of the quadratic diophantine equations with other special numbers.

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