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Intrinsic Solutions to the Biquadratic Diophantine Equations $x^4 - y^4 = 65(z^2 - w^2)p^2$

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Abstract: The Biquadratic Diophantine equation with five unknowns has non-zero unique intrinsic integral solutions obtained. The recurrence relation between the solutions is also obtained.
Keywords: Biquadratic Diophantine equation, Pell equation, integral solution.

I. INTRODUCTION

The reigning princess of mathematics is number theory. For Diophantine problems, when the number of equations is less than the number of unknowns, the objective is to find numbers that concurrently answer every equation. Due to its variety, the Biquadratic Diophantine equation provides an infinite field of study. The five unknown biquadratic equation $x^4 - y^4 = 65(z^2 - w^2)p^2$ is taken into consideration for its non-zero unique integer solutions. The recurrence relation is also found.

II. METHOD OF ANALYSIS

The non-zero distinct integral solution of the homogeneous Biquadratic Diophantine equations with five unknowns is

$$x^4 - y^4 = 65(z^2 - w^2)p^2 \tag{1}$$

Introducing the linear transformations

$$x = u + v, y = u - v, z = 2u + v, w = 2u - v \tag{2}$$

Equation (1) becomes

$$u^2 + v^2 = 65p^2 \tag{3}$$

Equation (3) can be solved in the following two methods.

A. Method I

The non-zero distinct integer solutions of the equation (3) are derived in the following three patterns.

1) Pattern I

Assume

$$65 = (8 + i)(8 - i) \tag{4}$$

and

$$p = a^2 + b^2 = (a + ib)(a - ib) \tag{5}$$

Using (4) and (5) in (3) and employing the method of factorization, we get,

$$(u + iv)(u - iv) = (8 + i)(8 - i)(a + ib)^2(a - ib)^2$$

Equating the like factors, we get,

$$(u + iv) = (8 + i)(a + ib)^2$$

$$(u - iv) = (8 - i)(a - ib)^2$$

Equating real and imaginary part, we get,

$$u = 8a^2 - 8b^2 - 2ab$$

$$v = a^2 - b^2 + 16ab$$

The non-zero unique integer solution to equation (2) after substituting u and v are:

$$x = x(a,b) = 9a^2 - 9b^2 + 14ab$$
$$y = y(a,b) = 7a^2 - 7b^2 - 18ab$$
$$z = z(a,b) = 17a^2 - 17b^2 + 12ab$$
$$w = w(a,b) = 15a^2 - 15b^2 - 20ab$$
$$p = p(a,b) = a^2 + b^2$$

Observations:

1. $x(a,1) + y(a,1) - w(a,1) - 2T_{3,a} - 7Gno_a - a \equiv 0(\text{mod}6)$.
2. $x(2,2) + 2z(1,1) + p(1,0)$ is a perfect square.
3. $2z(1,1)$ is a Nasty number.
4. $p(1,1)$ is a Palindrome number.
5. $3x(1,1)$ is a Harshad number.

2) Pattern II

The number 65 may also be expressed as follows:

$$65 = (1+8i)(1-8i) \tag{6}$$

By applying factorization and using (5) and (6) in (3), we obtain,

$$(u+iv)(u-iv) = (1+8i)(1-8i)(a+ib)^2(a-ib)^2$$

Applying the methodology of pattern I, the non-zero distinct integer solutions of (1) are

$$x = x(a,b) = 9a^2 - 9b^2 - 14ab$$
$$y = y(a,b) = -7a^2 + 7b^2 - 18ab$$
$$z = z(a,b) = 10a^2 - 10b^2 - 30ab$$
$$w = w(a,b) = -6a^2 + 6b^2 - 34ab$$
$$p = p(a,b) = a^2 + b^2$$

Observations:

1. $x(a,1) + y(a,1) - w(a,1) - Star_a - 2T_{3,a} - 2T_{3,a} - 4Gno_n \equiv 0(\text{mod}5)$.
2. $10p(1,1) - w(1,1)$ is a Nasty number.
3. $p(6,4) + z(1,1)$ is a Happy number.
4. $5p(1,1) + 3y(1,-1)$ is a perfect cube.
5. $w(1,-1) + z(1,1)$ is a perfect square.

3) Pattern III

Rewriting equation (3) as

$$1 * u^2 = 65p^2 - v^2 \tag{7}$$

Assume

$$u = 65a^2 - b^2 = (\sqrt{65}a + b)(\sqrt{65}a - b) \tag{8}$$

Write 1 as,

$$1=\left(\sqrt{65}+8\right)\left(\sqrt{65}-8\right) \tag{9}$$

Using (8) and (9) in (7) and employing the method of factorization, we get,

$$\left(\sqrt{65}+8\right)\left(\sqrt{65}-8\right)\left(\sqrt{65}a+b\right)^2\left(\sqrt{65}a-b\right)^2=\left(\sqrt{65}p+v\right)\left(\sqrt{65}p-v\right)$$

Equating the like factors, we get,

$$\left(\sqrt{65}+8\right)\left(\sqrt{65}a+b\right)^2=\left(\sqrt{65}p+v\right)$$

$$\left(\sqrt{65}-8\right)\left(\sqrt{65}a-b\right)^2=\left(\sqrt{65}p-v\right)$$

Equating rational and irrational part, we get,

$$p=65a^2+b^2+16ab$$

$$v=520a^2+8b^2+130ab$$

The non-zero unique integer solution to equation (2) after substituting u and v are:

$$x=x(a,b)=585a^2+7b^2+130ab$$

$$y=y(a,b)=-455a^2-9b^2-130ab$$

$$z=z(a,b)=650a^2+6b^2+130ab$$

$$w=w(a,b)=-390a^2-10b^2-130ab$$

$$p=p(a,b)=65a^2+b^2+16ab$$

Observations:

1. $x(1,b)+y(1,b)+z(1,b)-T_{10,b}-66Gno_b-b\equiv 0(\bmod 586)$
2. $z(1,1)-x(1,1)$ is a perfect cube.
3. $z(1,1)+w(1,1)$ is a perfect square.
4. $8p(1,1)+w(1,1)$ is a Niven number.
5. $p(1,1)$ is a Happy number.

B. *Method II*

Here a new method is used to solve (3) in order to require the integer solutions of (1). Let $u_0=8v, p_0=v$ be the least positive integer solution of (3). To derive the remaining solutions for equation (3), the Pell equation taken into consideration.

$$u^2=65p^2+1$$

whose initial solution

$$u_n=\frac{1}{2}f_n$$

$$p_n=\frac{1}{2\sqrt{65}}g_n$$

where

$$f_n=(129+16\sqrt{65})^{n+1}+(129-16\sqrt{65})^{n+1} \text{ and}$$

$$g_n=(129+16\sqrt{65})^{n+1}-(129-16\sqrt{65})^{n+1}$$

Brahmaguptalemma is applied between (u_0, p_0) and (u_n, p_n) to yield the series of non-zero unique integer solutions of(1).

$$x_{n+1} = v\{\frac{1}{2}[(129+16\sqrt{65})^{n+1}(8+\sqrt{65})+(129-16\sqrt{65})^{n+1}(8-\sqrt{65})]+1\}$$

$$y_{n+1} = v\{\frac{1}{2}[(129+16\sqrt{65})^{n+1}(8+\sqrt{65})+(129-16\sqrt{65})^{n+1}(8-\sqrt{65})]-1\}$$

$$z_{n+1} = v\{[(129+16\sqrt{65})^{n+1}(8+\sqrt{65})+(129-16\sqrt{65})^{n+1}(8-\sqrt{65})]+1\}$$

$$w_{n+1} = v\{[(129+16\sqrt{65})^{n+1}(8+\sqrt{65})+(129-16\sqrt{65})^{n+1}(8-\sqrt{65})]-1\}$$

$$p_{n+1} = \frac{1}{2\sqrt{65}}v[(129+16\sqrt{65})^{n+1}(\sqrt{65}-8)+(129-16\sqrt{65})^{n+1}(\sqrt{65}-8)]$$

The solutions to equation (1) satisfy the following recurrence relations

$$x_{n+1} - 258x_{n+2} + x_{n+3} = -256v$$

$$y_{n+1} - 258y_{n+2} + y_{n+3} = 256v$$

$$z_{n+1} - 258z_{n+2} + z_{n+3} = -256v$$

$$w_{n+1} - 258w_{n+2} + w_{n+3} = 256v$$

III. CONCLUSION

Other non-zero unique integer solutions to the Biquadratic equations with multiple variables under consideration can be found**.** To conclude, one can search for biquadratic Diophantine equations for various numbers with the corresponding properties.

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