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Neutrosophic Quotient Spaces and the First Isomorphism Theorem for Topological Groups

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Abstract: In this paper, the concept of Neutrosophic Topological Groups (NTGs) is introduced and studied. By combining Smarandache's Neutrosophic logic and topological groups theory [1], the NTGs are defined. Since classical set theories and classical logic are not enough to deal with indeterminate data, neutrosophic sets have been used for this work. In the neutrosophic sets, each element has a truth-membership function, indeterminacy-membership function and false-membership function. The continuity of multiplication and inversion operations for the NTGs are defined. Moreover, the NTGs quotient spaces which are induced by normal subgroups are constructed. The main target of this study is to prove the continuity and openness of the projection map. Therefore, this paper ends with formulating and proving a Neutrosophic version of the First Isomorphism Theorem for topological groups.

Keywords: Neutrosophic Sets; Neutrosophic Topology; Topological Groups; Quotient Spaces; Homomorphisms; Isomorphism Theorems.

I. INTRODUCTION

The field of abstract algebra and topology has seen significant changes in the past century. The main characteristic of this development has been the constant need to generalize concepts in order to operate with increasingly complex and uncertain systems. The concept of neutrosophic logic, presented by Florentin Smarandache, is a giant step in this direction. By introducing an independent degree of indeterminacy, it fundamentally extends both fuzzy logic and intuitionistic fuzzy logic [1]. This allows one to work with real-world systems with their associated uncertainties, inconsistencies, and incomplete information.

Topological groups have been widely studied in fuzzy and intuitionistic fuzzy settings. But the algebraic and topological properties of groups with neutrosophic topologies still present a promising area of research. The connection between abstract algebra and the neutrosophic topology poses additional challenges since group operations should interact with the truth, indeterminacy, and falsehood parameters of open sets. In this research paper, we aim to describe the properties of Neutrosophic Topological Groups (NTGs) in detail. We will also study the properties of translation mappings, define neutrosophic quotient topologies, and investigate neutrosophic continuous homomorphisms in depth.

The mathematical foundation for handling uncertainty began with Zadeh's introduction of fuzzy sets [2], which provided a vital framework for modeling degrees of membership. Taking this idea further, Chang introduced fuzzy topological spaces [3], applying the concept of uncertainty to mathematical spaces. A few decades later, Atanassov took this a step further with intuitionistic fuzzy sets [4], adding a specific non-membership degree into the mix. But both of these theories struggled with one major issue of pure indeterminacy. When real-world information is inconsistent or incomplete, those earlier models simply couldn't capture the indeterminacy independently. That changed when Smarandache introduced neutrosophic logic and sets [1]. By adding an independent indeterminacy function, he completely shifted how we handle uncertain data. Building directly on that idea, Salama and Alblawi worked out the formal axioms for neutrosophic topological spaces [5]. They established the strict rules needed to define continuity and open sets in this new environment.

On the abstract algebra side, the classical theory of topological groups [6], [7] has been around for a long time. Mathematicians like Husain and Pontryagin did a lot of the foundational work, showing how continuous spatial transformations can seamlessly merge with group operations. Recently, we've seen a strong push to bring neutrosophic logic into these rigid algebraic structures. Ali and his team generalized neutrosophic rings and fields [8]. Agboola and his colleagues also dug deeper into these neutrosophic algebraic frameworks [9]. Around the same time, Bera and Mahapatra started exploring how neutrosophic environments behave in soft topological spaces [10].

Yet, even with all this recent progress, there is still a noticeable gap in the literature. We lack a rigorous, dedicated study focused strictly on Neutrosophic Topological Groups (NTGs). Specifically, no one has really broken down how neutrosophic open sets behave during continuous translations or canonical projections to form functional quotient spaces. Understanding this structural behavior is absolutely essential if we want to extend the classical Isomorphism Theorems into the neutrosophic domain. Filling that gap is exactly what motivates the work in this paper.

II. PRELIMINARIES AND MATHEMATICAL DEFINITIONS

Before stating and proving the theorems, we need to introduce the definitions that describe the properties of Neutrosophic Spaces and Mappings.

Definition 2.1 (Neutrosophic Set) - Let X be a non-empty universal space. A neutrosophic set A in X is defined by three independent membership functions: the truth-membership function $T_A(x)$, the indeterminacy-membership function $I_A(x)$, and the falsehood-membership function $F_A(x)$. Here, T_A , I_A , and F_A are mappings from X to the standard unit interval $[0,1]$. Formally, the set is denoted as:

$$A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X \}$$

Definition 2.2 (Neutrosophic Topology) - A neutrosophic topology on a non-empty set X is a family τ of neutrosophic sets in X that strictly satisfies three axioms:

1. The absolute neutrosophic empty set (0_N) and the absolute universal set (1_N) must belong to τ .
2. The intersection of any two neutrosophic sets in τ remains in τ .
3. The union of any arbitrary collection of neutrosophic sets in τ belongs to τ . The pair (X, τ) is then called a Neutrosophic Topological Space (NTS).

Definition 2.3 (Neutrosophic Topological Group) - Let G be a group, and let τ be a neutrosophic topology defined on the underlying set of G . The pair (G, τ) is classified as a Neutrosophic Topological Group (NTG) if and only if the fundamental group operations are neutrosophic continuous. Specifically:

1. The multiplication map $m: G \times G \rightarrow G$ defined by $m(x, y) = xy$ is neutrosophic continuous.
2. The inversion map $inv: G \rightarrow G$ defined by $inv(x) = x^{-1}$ is neutrosophic continuous.

Theorem 2.1 (Translation Invariance) - Let (G, τ) be a Neutrosophic Topological Group, and let $a \in G$ be an arbitrary element. Then the left translation mapping $L_a: G \rightarrow G$ defined by $L_a(x) = ax$, and the right translation mapping $R_a: G \rightarrow G$ defined by $R_a(x) = xa$, are both neutrosophic homeomorphisms.

Proof - To mathematically establish that the left translation L_a is a neutrosophic homeomorphism, we must prove three properties: it is bijective, neutrosophic continuous, and a neutrosophic open mapping.

Let $x, y \in G$. If $L_a(x) = L_a(y)$, then $ax = ay$. By the cancellation law in groups, $x = y$, proving injectivity. For any $y \in G$, there exists an element $x = a^{-1}y \in G$ such that $L_a(x) = a(a^{-1}y) = y$, proving surjectivity. Thus, L_a is a crisp bijection.

Let $V \in \tau$ be an arbitrary neutrosophic open set in G . We must demonstrate that its pre-image $L_a^{-1}(V) \in \tau$.

Since (G, τ) is a Neutrosophic Topological Group (NTG), the multiplication map $m: G \times G \rightarrow G$, defined by $m(x, y) = xy$, is neutrosophic continuous. This implies that $m^{-1}(V)$ is a neutrosophic open set in the product topology on $G \times G$.

We can define an insertion mapping $j_a: G \rightarrow G \times G$ as $j_a(x) = (a, x)$. In any topological space, fixing a single coordinate (a topological slice) preserves continuity.

Therefore, the composition $m \circ j_a$ is neutrosophic continuous.

Notice that $(m \circ j_a)(x) = m(a, x) = ax = L_a(x)$.

Since L_a is equivalent to the continuous composition $m \circ j_a$, it unequivocally follows that the pre-image $L_a^{-1}(V)$ belongs to τ . Hence, L_a is neutrosophic continuous.

This is where we evaluate the exact neutrosophic behavior rather than relying solely on abstract compositions.

Let $U \in \tau$ be a neutrosophic open set. We must prove that its direct image $L_a(U)$ is also in τ .

Let us evaluate the truth-membership function of $L_a(U)$ at an arbitrary point $y \in G$. By the standard definition of the image of a neutrosophic set:

$$T_{L_a(U)}(y) = \bigvee_{x \in L_a^{-1}(y)} T_U(x)$$

Because L_a is a rigorous bijection (as proved above), the pre-image set $L_a^{-1}(y)$ contains one and only one element: $x = a^{-1}y$. The supremum over a singleton set is simply the value at that element. Thus, the equation simplifies exactly to:

$$T_{L_a(U)}(y) = T_U(a^{-1}y)$$

By applying identical logic to the indeterminacy and falsehood membership functions, we obtain:

$$I_{L_a(U)}(y) = I_U(a^{-1}y)$$

$$F_{L_a(U)}(y) = F_U(a^{-1}y)$$

Now, observe the right-hand side of these equations. The values $T_U(a^{-1}y)$, $I_U(a^{-1}y)$, and $F_U(a^{-1}y)$ perfectly correspond to the membership functions of the pre-image of U under the mapping L_a^{-1} .

Mathematically, this establishes a structural equality:

$$L_a(U) = (L_a^{-1})^{-1}(U)$$

Since a^{-1} is just another element in G , the mapping $L_{a^{-1}}$ is neutrosophic continuous by the exact same topological logic established in Step 2. Because L_a^{-1} is continuous and $U \in \tau$, the pre-image $(L_a^{-1})^{-1}(U)$ must be open in τ .

Substituting the equality, we get $L_a(U) \in \tau$.

Since L_a is a bijective, neutrosophic continuous mapping that sends neutrosophic open sets strictly to neutrosophic open sets, L_a is a neutrosophic homeomorphism. The proof for the right translation R_a follows an identical, symmetric trajectory.

III. NEUTROSOPHIC QUOTIENT TOPOLOGICAL SPACES

To define quotient spaces properly, we must closely observe how a neutrosophic topology behaves when it collapses under a canonical projection mapping associated with a normal subgroup.

Theorem 3.1 (Canonical Projection as an Open Mapping) - *Let (G, τ) be a Neutrosophic Topological Group, and let H be a normal subgroup of G . The canonical projection $p: (G, \tau) \rightarrow (G/H, \tau_{G/H})$ defined by $p(x) = xH$ is a neutrosophic open mapping.*

Proof - To mathematically establish that the canonical projection p is a neutrosophic open mapping, we must demonstrate that for any arbitrary neutrosophic open set $A \in \tau$, its direct image $p(A)$ belongs to the neutrosophic quotient topology $\tau_{G/H}$.

By the fundamental definition of the quotient topology, a neutrosophic set U resides in $\tau_{G/H}$ if and only if its pre-image $p^{-1}(U)$ is open in the parent topology τ .

Thus, to show that $p(A) \in \tau_{G/H}$, our objective strictly reduces to proving that the neutrosophic set $p^{-1}(p(A)) \in \tau$.

This is where we evaluate the exact neutrosophic behavior of the projection mapping.

Let us evaluate the truth-membership function of $p^{-1}(p(A))$ at an arbitrary element $x \in G$. By definition of the neutrosophic pre-image: $T_{p^{-1}(p(A))}(x) = T_{p(A)}(p(x))$

Applying the formal definition of the neutrosophic direct image, this expands to a supremum over the pre-image of the coset:

$$T_{p(A)}(p(x)) = \bigvee_{z \in p^{-1}(p(x))} T_A(z)$$

Since p is the canonical projection mapping an element to its left coset, $p(x) = xH$. The complete pre-image of this specific coset in the group G is precisely the coset itself.

Therefore, any element $z \in p^{-1}(p(x))$ must inherently take the form $z = xh$ for some $h \in H$.

This algebraic substitution allows us to rewrite the supremum over the subgroup H :

$$T_{p^{-1}(p(A))}(x) = \bigvee_{h \in H} T_A(xh)$$

Utilizing the right translation operator $R_h(x) = xh$, this expression perfectly transforms into:

$$T_{p^{-1}(p(A))}(x) = \bigvee_{h \in H} T_A(R_h(x))$$

By applying identical logic to the indeterminacy and falsehood membership functions, we naturally derive:

$$I_{p^{-1}(p(A))}(x) = \bigvee_{h \in H} I_A(R_h(x))$$

$$F_{p^{-1}(p(A))}(x) = \bigwedge_{h \in H} F_A(R_h(x))$$

These simultaneous equations mathematically confirm that the set $p^{-1}(p(A))$ is structurally identical to the neutrosophic union of right translations of A by elements of the subgroup H . Formally, this establishes the functional equivalence:

$$p^{-1}(p(A)) = \bigcup_{h \in H} (R_h)^{-1}(A)$$

From Theorem 2.1, we established that right translations in a Neutrosophic Topological Group are robust neutrosophic homeomorphisms. Consequently, since A is a neutrosophic open set, its pre-image under any continuous right translation $(R_h)^{-1}(A)$ is unequivocally open in τ .

According to the fundamental axioms of neutrosophic topology, an arbitrary union of neutrosophic open sets remains a neutrosophic open set. Thus, the infinite union

$$\bigcup_{h \in H} (R_h)^{-1}(A)$$

belongs to τ , which definitively proves that $p^{-1}(p(A)) \in \tau$.

Because its pre-image is mathematically open in G , it inherently follows that $p(A) \in \tau_{G/H}$, confirming that the canonical projection p is a rigorous neutrosophic open mapping.

IV. ISOMORPHISM THEOREMS IN NTGs

The Isomorphism Theorems are cornerstones of abstract algebra. Extending them to the neutrosophic realm requires delicate handling of open and continuous mappings.

Theorem 4.1 (Neutrosophic First Isomorphism Theorem) - Let (G, τ_G) and (K, τ_K) be two Neutrosophic Topological Groups. Suppose $f: G \rightarrow K$ is a surjective group homomorphism that is both neutrosophic continuous and neutrosophic open, with its kernel denoted as H . Then H is a normal subgroup of G , and the quotient neutrosophic topological group $(G/H, \tau_{G/H})$ is neutrosophic isomorphic to (K, τ_K) .

Proof - To mathematically establish that the quotient group is neutrosophic isomorphic to K , we must prove the existence of a well-defined algebraic isomorphism that simultaneously acts as a neutrosophic homeomorphism (both neutrosophic continuous and neutrosophic open).

From classical abstract algebra, because f is a surjective group homomorphism with its kernel defined as H , we immediately deduce that H inherently forms a normal subgroup of G .

The fundamental theorem of algebraic homomorphisms guarantees the existence of a well-defined, bijective algebraic group isomorphism $\phi: G/H \rightarrow K$, defined explicitly by $\phi(xH) = f(x)$.

The structural mapping relationship can be elegantly expressed as a composition: $f = \phi \circ p$, where $p: G \rightarrow G/H$ is the canonical projection mapping.

Let $V \in \tau_K$ be an arbitrary neutrosophic open set in K . To prove ϕ is neutrosophic continuous, we must strictly show that its pre-image $\phi^{-1}(V)$ belongs to $\tau_{G/H}$.

By the formal definition of the neutrosophic quotient topology $\tau_{G/H}$, the set $\phi^{-1}(V)$ is open if and only if its pre-image under the canonical projection p is open in G .

Let us mathematically examine this pre-image $p^{-1}(\phi^{-1}(V))$. Using the standard properties of function composition, this equates to: $p^{-1}(\phi^{-1}(V)) = (\phi \circ p)^{-1}(V)$

Substituting the fundamental mapping relationship $f = \phi \circ p$, we obtain: $p^{-1}(\phi^{-1}(V)) = f^{-1}(V)$

We are given the critical premise that f is a neutrosophic continuous homomorphism. Therefore, the pre-image of any open set V under f must be strictly open in G . Thus, $f^{-1}(V) \in \tau_G$.

This completely satisfies the condition for the quotient topology, irrevocably yielding $\phi^{-1}(V) \in \tau_{G/H}$.

Hence, ϕ is a neutrosophic continuous mapping.

Next, let $U \in \tau_{G/H}$ be an arbitrary neutrosophic open set in the quotient space. We must rigorously demonstrate that its direct image $\phi(U)$ is neutrosophic open in K .

By the inherent definition of the quotient topology, $U \in \tau_{G/H}$ unconditionally implies that $p^{-1}(U) \in \tau_G$.

We operate under the given assumption that f is a neutrosophic open mapping. Therefore, the image of the open set $p^{-1}(U)$ under f must logically be open in K . Mathematically, this dictates: $f(p^{-1}(U)) \in \tau_K$

Recalling our composition equation $f = \phi \circ p$, we can structurally rewrite this image as:

$$f(p^{-1}(U)) = \phi(p(p^{-1}(U)))$$

Since the canonical projection p is a structurally surjective mapping, applying p to the exact pre-image of U yields exactly U itself. That is, $p(p^{-1}(U)) = U$.

Substituting this fundamental identity back simplifies our expression perfectly to: $f(p^{-1}(U)) = \phi(U)$

Since we have already established that $f(p^{-1}(U)) \in \tau_K$, it unequivocally follows that $\phi(U) \in \tau_K$.

Hence, ϕ strictly maps neutrosophic open sets to neutrosophic open sets, making it a neutrosophic open mapping.

Since $\phi: G/H \rightarrow K$ is a mathematically rigorous bijective group homomorphism that simultaneously satisfies the exact conditions of being neutrosophic continuous and neutrosophic open, ϕ is a true neutrosophic homeomorphism.

Therefore, $(G/H, \tau_{G/H})$ is neutrosophic isomorphic to (K, τ_K) .

V. CONCLUSION

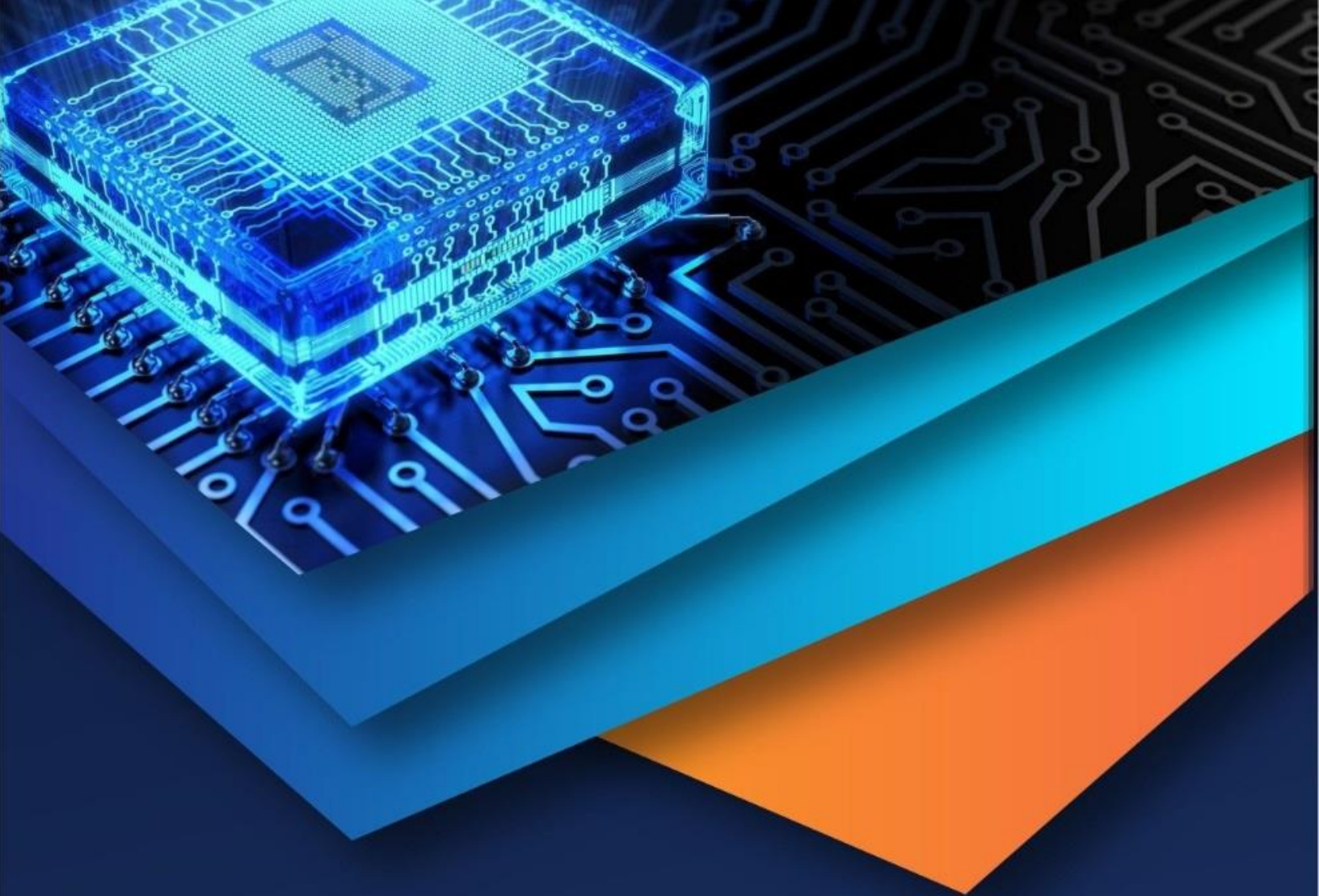
In this study, we were able to generalize the classical and fundamental notions of topological groups to the very flexible neutrosophic environment. We proved that left, and right group translations are still homeomorphisms in case of Neutrosophic Topological Groups. We were able to define the so-called neutrosophic quotient topology and proved that the corresponding canonical projection is an open map. This allowed us to formulate and prove the Neutrosophic First Isomorphism Theorem. Our results create the fertile ground for future research as the researchers can follow our lead, and try to establish the basic neutrosophic separation axioms and the various forms of compactness, and investigate their behavior in different quotient spaces.

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