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Numerical Approximation of Poisson Equation Using the Finite Difference Method

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Abstract: In this paper, we implement the Finite Difference Method to approximate the homogeneous form of the Poisson equation. The Poisson equation is discretized using the central difference approximation of the second derivative and the grid function is determined by the five point method approximates the exact solution of the Poisson equation. The finite difference approximation is consistent and convergent. The method of solving the numerical approximation of Poisson equation is implemented using Python Programming.
Keywords: Poisson equation, Finite Difference method, Discrete Grid, Numerical Approximation, Mesh Points, Matrix form, Python.

I. INTRODUCTION

Partial Differential Equations arise in the study of many branches of Applied Mathematics, for instance, in the field of fluid dynamics, heat transfer, boundary layer flow, elasticity, quantum mechanics and electromagnetic theory. The analytical method of these equations is a rather involved process and requires applications of advanced mathematical methods. In most of the cases, it is easier to develop approximate solutions by numerical methods. Several numerical methods have been proposed for the solution of partial differential equations, the method of finite difference is most commonly used. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations.

H.Bennour and M.S.Said have investigated the solution of Poisson equation in n-dimensional domain with Dirichlet boundary conditions to establish the existence, uniqueness and regularity of the solution [1]. Mohammad Mehdi Mazarei and Azim Aminataei have introduced a new way for numerical solution of Poisson’s partial differential equation by a special combination between logarithmic and multi-quadric radial basis function [2]. Peng Guo studied the two-dimensional Poisson equation with Dirichlet boundary conditions. Five point difference method and Chebyshev spectral method is used to solve the corresponding two-dimensional Poisson equation [3]. James R. Nagel has proposed the practical application of multiple dielectrics, conductive materials and magnetostatics using the finite difference method from Poisson equation [4]. Iman Shojaei et al., have developed the solution of a governing equation on an arbitrary domain is sought through a geometrical transformation from the rectangular domain into the original domain using conformal mapping. They have proved that conformal mapping preserves the Laplace and Poisson equation which are used in engineering problem [5]. Mohammad Mehdi Mazarei and Azim Aminataei have proposed that the transformation of Poisson’s equation into the polar coordinate can achieve a better accuracy than the direct radial basis function network method and the indirect radial basis function network method on the Cartesian coordinates [6].

Benyam Mebrate and Purnachandra Rao Koya have introduced Microsoft Office Excel worksheet implementations of numerical methods for solving Poisson’s equation in two dimensions with Dirichlet’s boundary conditions. They have used finite difference method and finite element method and the numerical solutions obtained by these two methods are also compared with each other graphically in two and three dimensions [7]. Mohammad Asif Zaman has presented a comprehensive discussion on how to build a finite difference matrix solver that can solve the Poisson equation for arbitrary geometry and boundary conditions [8]. D.J. Evans has introduced a rhombic region to solve the Poisson equation using skew rectangular coordinates by the successive block over-relaxation method [9]. Genet Mekonnen Assefa and Lemi Guta have showed that finite difference method for two-dimensional Poisson equation with non-uniform mesh is not sufficiently accurate than finite difference method for two-dimensional Poisson equation with uniform mesh size [10]. Mohammad Aslefallah and David Rostamy have presented a numerical scheme for solving fractional Poisson equation. The method is used to find the numerical solutions of these equations based on the Grunwald estimates for Riemann-Liouville fractional derivative [11]. This paper proposes the numerical approximation of the Poisson equation using finite difference method. The paper is organized as follows: Section II presents the Finite Difference Method, Section III discusses the Poisson Equation, Section IV focuses on Implementation and Results, Section V analyses on Consistency and Convergence of the Poisson equation and finally the Conclusion is presented in Section VI.

II. FINITE DIFFERENCE METHOD

This is a numerical technique to solve a partial differential equation. Here we approximate second order partial derivatives using finite differences. Consider a two-dimensional region where the function $f(x,y)$ is defined. The domain is split into regular rectangular grids. The points of intersection of these points are called mesh points, grid points or nodal points. The approximation is made discrete values of the independent variables and approximation is implemented through a computer program. The finite difference method replaces all partial derivatives and other terms in the partial derivatives by means of their finite difference approximations. After some modification, a finite difference scheme is created from which the approximate solution is obtained [12].

To find the solution of the function $f(x,y)$ on the region, we divide the solution region into equal rectangles or meshes. We now construct a finite difference equation to represent the given equation and boundary conditions [13]. This will help us to calculate the values of $f(x,y)$ at the nodes of the rectangles in the region. Here we directly consider the difference equation as

$$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 f_{i,j}] \dots\dots\dots (1)$$

Equation (1) shows that the value of u at any point is the mean of its values at the four neighbouring points. This is called the standard five point formula. Solution of elliptic partial differential equations is over closed regions on which boundary values are given. The boundary values determine the solution of the partial differential equation in the interior of the region. The two most widely used elliptic partial differential equations are Laplace equation and Poisson equation [14].

III. POISSON EQUATION

The general two dimensional Poisson's equation is of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), (x,y) \in \Omega = (0,1) \times (0,1) \dots\dots\dots (2)$$

with boundary conditions $U(x,y) = g(x,y)$ $where$ $(x,y) \in \partial\Omega$-boundary

The region $\Omega = (0,1) \times (0,1)$ is discretized into a uniform mesh Ω_h. In the x and y directions into N steps giving step size of $h = \frac{1-0}{N}$ where

$$\left.\begin{aligned} x[i] &= 0 + ih, \quad i = 0,1,2, \dots\dots\dots, N \\ and\ x[j] &= 0 + jh, \quad j = 0,1,2, \dots\dots\dots, N \end{aligned}\right\} \dots\dots\dots (3)$$

The Poisson equation is discretized using δ_x^2 is the central difference approximation of the second derivative in the x direction

$$\delta_x^2 = \frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] \dots\dots\dots (4)$$

and δ_y^2 is the central difference approximation of the second derivative in the y direction

$$\delta_y^2 = \frac{1}{h^2}[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}] \dots\dots\dots (5)$$

This gives the Poisson difference equation,

$$-[\delta_x^2 u_{i,j} + \delta_y^2 u_{i,j}] = f_{i,j}, \quad (x_i, y_j) \in \Omega_h \dots\dots\dots (6)$$

$$u_{i,j} = g_{i,j} \quad , (x_i, y_j) \in \partial\Omega_h$$

where $u_{i,j}$ is the numerical approximation of U at x_i and $, y_j$.

Expanding the Poisson difference equation gives the five point method,

$$-[u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i,j+1} + u_{i+1,j}] = h^2 f_{i,j} \dots\dots\dots (7)$$

for $i = 1,2, \dots\dots\dots, N-1$ and $j = 1,2, \dots\dots\dots, N-1$.

Equation (7) can be written as $\nabla_h^2 u_{i,j} = f_{i,j} \dots\dots\dots (8)$

IV. IMPLEMENTATION AND RESULTS

We will implement a finite difference method to approximate the Poisson equation by using Python programming. Let us consider the homogeneous second order form of the Poisson's equation where $f(x, y) = 0$.

∴ The homogeneous second order Poisson equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ldots\ldots\ldots (9)$ with boundary conditions

$$u(x, 0) = \sin(2\pi x), \quad 0 \le x \le 1, \quad lower,$$
$$u(x, 1) = \sin(2\pi x), \quad 0 \le x \le 1, \quad upper,$$
$$u(0, y) = 2\sin(2\pi y), \quad 0 \le y \le 1, \quad left,$$
$$u(1, y) = 2\sin(2\pi y), \quad 0 \le y \le 1, \quad right.$$

The region $\Omega = (0,1) \times (0,1)$ is discretized into a uniform mesh Ω_h. In the x and y directions into $N = 10$ steps giving step size of

$$h = \frac{1-0}{10} = 0.1 \quad \text{where}$$
$$x[i] = 0 + ih, \quad i = 0,1,2, \ldots\ldots\ldots, 10$$
$$and\ x[j] = 0 + jh, \quad j = 0,1,2, \ldots\ldots\ldots, 10$$

The Figure 1.below shows the discrete grid points for $N = 10$, the known boundary conditions (green colour) and the unknown values (red colour) of the Poisson equation.

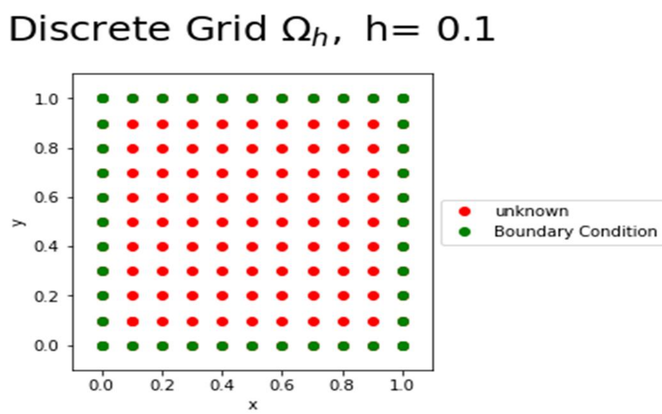

Figure 1.Discrete grid points for $N = 10$ of the Poisson equation.

The discrete boundary conditions of Equation (9) are

$$u_{i0} = u(i, 0) = \sin(2\pi x[i]), \quad \text{for i} = 0,1, \ldots\ldots\ldots 10, \quad lower,$$
$$u_{iN} = u(i, \text{N}) = \sin(2\pi x[i]), \quad \text{for i} = 0,1, \ldots\ldots\ldots 10, \quad upper,$$
$$u_{0j} = u(0, \text{j}) = 2\sin(2\pi y[j]), \quad \text{for i} = 0,1, \ldots\ldots\ldots 10, \quad left,$$
$$u_{Nj} = u(N, j) = 2\sin(2\pi y[j]), \quad \text{for i} = 0,1, \ldots\ldots\ldots 10, \quad right.$$

The following Figure plots the boundary values of $u(i, j)$

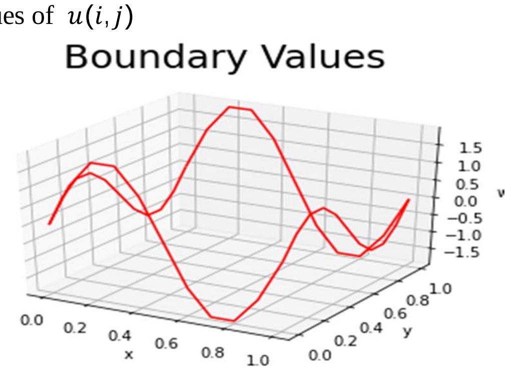

Figure 2.Plots of the boundary values of $u(i, j)$.

Equation (8) can be written as a system of $(N-1) \times (N-1)$ can be arranged in matrix form

$$A\boldsymbol{w} = \boldsymbol{r} \ldots\ldots\ldots (10)$$

where A is an $(N-1)^2 \times (N-1)^2$ matrix made up of the following block tridiagonal form

$$\begin{bmatrix} T & I & 0 & 0 & . & . & . \\ I & T & I & 0 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & 0 & I & T & I \\ . & . & . & . & 0 & I & T \end{bmatrix}$$

where I denote an $(N-1) \times (N-1)$ identity matrix and T is the tridiagonal matrix of the form

$$T = \begin{bmatrix} -4 & 1 & 0 & 0 & . & . & . \\ 1 & -4 & 1 & 0 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & 0 & 1 & -4 & 1 \\ . & . & . & . & 0 & 1 & -4 \end{bmatrix}$$

The plot below shows the matrix A and its inverse A^{-1} *as* a colour plot.

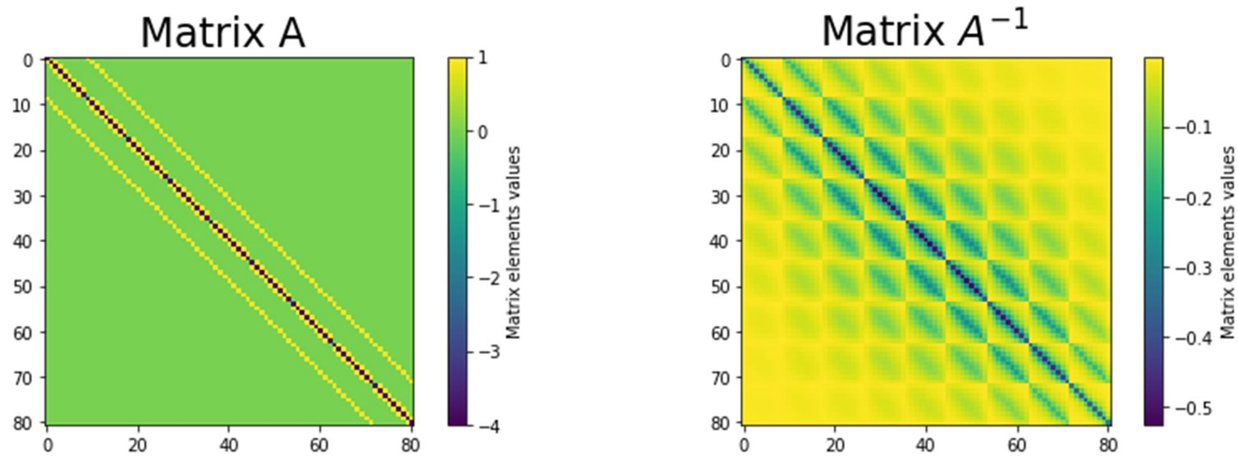

Figure 3.Matrix A and Matrix A^{-1}

The vector $\boldsymbol{w}$ is of length $(N-1) \times (N-1)$ which is made up of $(N-1)$ subvectors $\boldsymbol{w}_j$ of length $(N-1)$ of the form

$$\boldsymbol{w}_j = \begin{bmatrix} w_{1j} \\ w_{2j} \\ . \\ . \\ w_{N-2j} \\ w_{N-1j} \end{bmatrix}$$

The vector $\boldsymbol{r}$ is of length $(N-1) \times (N-1)$ which is made up of $(N-1)$ subvectors of the form $\boldsymbol{r}_j = -h^2 \boldsymbol{f}_j - \boldsymbol{bx}_j - \boldsymbol{by}_j$, where $\boldsymbol{bx}_j$ is the vector of left and right boundary conditions for $j = 1,2, \ldots\ldots\ldots, N-1$.

$$\boldsymbol{bx}_j = \begin{bmatrix} w_{0j} \\ 0 \\ . \\ . \\ 0 \\ w_{Nj} \end{bmatrix}$$

where $\boldsymbol{by}_j$ is the vector of the lower boundary condition for $j = 1$,

$$\boldsymbol{by}_1 = \begin{bmatrix} w_{10} \\ w_{20} \\ . \\ . \\ w_{N-20} \\ w_{N-10} \end{bmatrix}$$

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Upper boundary condition for $j = N-1$

$$\boldsymbol{by}_{N-1} = \begin{bmatrix} w_{1N} \\ w_{2N} \\ . \\ . \\ w_{N-2N} \\ w_{N-1N} \end{bmatrix}$$

$for\ \ j = 2,3, \dots \dots \dots, N-2, \boldsymbol{by}_j = 0, and\ \boldsymbol{f}_j = -\begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 0 \end{bmatrix} for\ j = 1,2, \dots \dots \dots, N-1.$

To solve the system for $\boldsymbol{w}$, invert the matrix A from equation (10), $A\boldsymbol{w} = \boldsymbol{r}$ such that

$$\boldsymbol{w} = A^{-1}\boldsymbol{r}.$$

Since $\boldsymbol{w}$ is a vector it has to be reshaped into grid form to plot. The following figure shows the numerical approximation of the homogeneous Poisson equation.

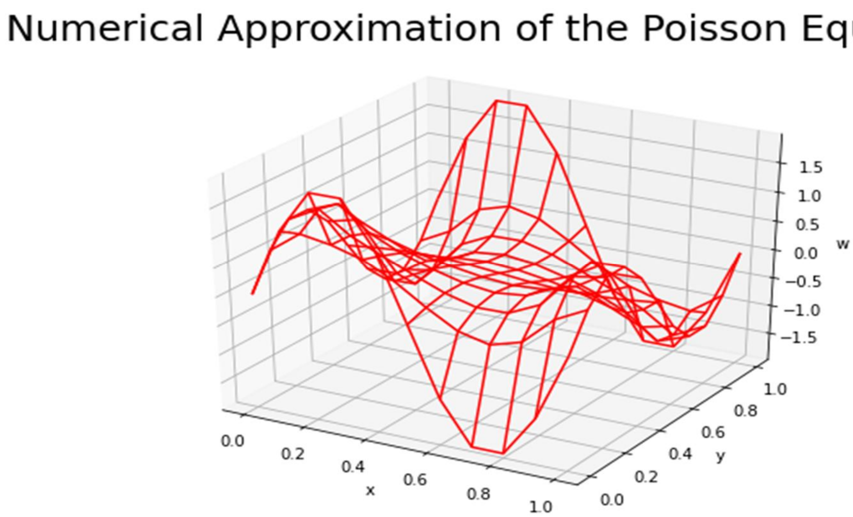

Figure 4.Numerical approximation of the homogeneous Poisson equation.

V. CONSISTENCY AND CONVERGENCE

Consistency and Convergence of the grid function determined by the five point method approximates the exact solution of the homogeneous Poisson equation.

Consistency: Let $\nabla_h^2(u) = -[u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i,j+1} + u_{i+1,j}] \dots \dots \dots (11)$

denote the finite difference approximation associated with the grid Ω_h having the mesh size h to a partial differential operator

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \dots \dots \dots (12)$$

defined on a simply connected open set $\Omega \in R^2$. For a given function $u \in C^{\infty}(\Omega)$, the truncation error of ∇_h^2 is

$$T_h(x) = [\nabla^2 - \nabla_h^2]u(x) \dots \dots \dots (13)$$

The approximation ∇_h^2 is consistent with ∇^2 if $\lim_{h\to 0}[T_h(x)] = 0$, for all$x \in D\ and\ all$ $u \in C^{\infty}(\Omega)$. The approximation is consistent to order p if $T_h(x) = O(h^p)$.

In other words, the method is consistent with the approximation of the Poisson equation.

Convergence: Let $\nabla_h^2 w(x_j) = f(x_j)$ be a finite difference approximation defined on a grid mesh size h to a partial differential equation $\nabla^2 U(x) = f(x)$ on a simply connected set $D \subset R^n$. Assume that $w(x,y) = U(x,y)$ at all points (x,y) on the boundary $\partial\Omega$. The finite difference scheme converges, if

$$\max_j |U(x_j) - w(x_j)| \to 0\ as\ h \to 0.$$

VI. CONCLUSION

We have implemented a finite difference method to approximate numerically to the second order homogeneous Poisson equation. The Poisson equation is discretized using finite difference method. By expanding the Poisson difference equation, we obtain the five point method equation. This equation is represented in matrix form to find the numerical solution of Poisson equation. Python programming is implemented to obtain the numerical solution of the Poisson equation with boundary conditions. Figure 1 shows the discrete grid points for $N = 10$ of the Poisson equation, Figure 2 shows the boundary values of the Poisson problem, Figure 3 shows the matrix form and its inverse matrix form and Figure 4 gives the numerical approximation of the Poisson equation. The grid function determined by the five point method approximates the exact solution of Poisson equation ensures the consistency and convergence of the solution.

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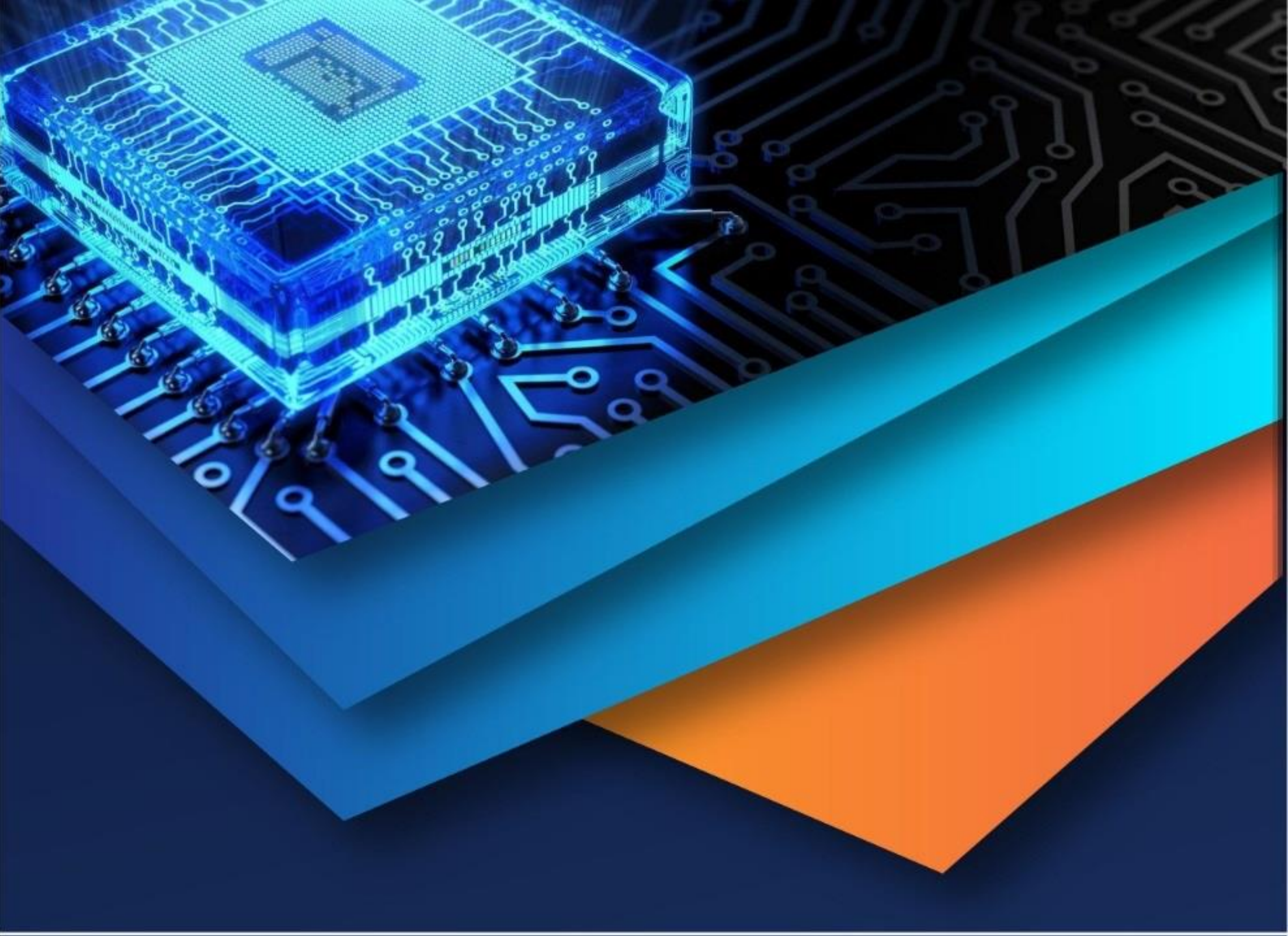

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