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Numerical Solution for the Heat Equation Using Forward Time Centered Space Difference Method

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Abstract: In this paper, the forward time centered space is applied to a simple problem involving one dimensional heat equation. Numerical solution to the heat equation using forward time centered space difference equation is obtained. The method of solving the problem is implemented by using Python Programming. We have discussed the local truncation error and stability analysis of explicit forward time centered space difference method of heat equation.
Keywords: Heat equation, Forward Time Centered space, Matrix equation, Boundary conditions, Truncation error, Stability analysis.

I. INTRODUCTION

The Heat diffusion equation is a parabolic partial differential equation which describes the heat distribution in a given region and provides the basic tool for heat conduction analysis. Analytical and Numerical methods have gained the interest of researchers for finding approximate solution to partial differential equations. Numerical Methods have applied to calculate the approximate solutions using Finite Difference Method.

Xiao-Jun Yang and Feng Gao have proposed a new technology for combining the variation of iterative method and an integral transforms for solution of diffusion and heat equation for the first time. The authors have found that the method is accurate and efficient in development of approximate solutions for the partial differential equations [1]. Alice Gorguis and Wai Kit Benny Chan have investigated a comparative study between the separation of variables and the Adomian method for heat equation. The study shows that Adomian has significant advantages and the method provides fast convergent series that gives exact solution for heat equation over the existing techniques [2]. Gerald W. Recktenwald provided a practical overview of numerical solutions to the heat equation using the finite difference method. The forward time centered space and the backward time centered space applied to a simple problem involving one dimensional differential heat equation. MATLAB codes were used to implement the differences between forward time centered space and backward time centered space [3].

Clint N.Dawson, Qiang DU and Todd F. Dupont have presented a domain decomposition algorithm for numerical solution of heat equation [4]. Abdulla-Al-Mamun et al. have obtained the analytical solution using MATLAB program by considering second order heat equation [5]. Jesus Martin-Vaqueno and Svajunas Sajavicius have demonstrated that the explicit forward time centered space scheme can be stable while some implicit methods such as Crank-Nicolson are unstable [6]. Hooshmandasl M.R., et al. have applied operational matrices of integration to get numerical solution of the one dimensional heat equation with Dirichlet boundary conditions. They have concluded that the use of Chebyshev wavelets is found to be accurate, simple and fast [7]. Tahrich N.A.Shahid et al. have investigated that modified difference equation in specific problems are more convenient for discussing the solution behavior including physical interpretation of accuracy, stability and consistency [8]. Jeffry Kusuma et al. have described a numerical solution for mathematical model of the transport equation in a simple rectangular box domain. Their mathematical model with various advection and diffusion parameters and boundary conditions is also solved numerically using finite difference forward time centered space method [9].

Wahida Zaman Loksar and Rama Sarkar have considered one dimensional heat equation as an initial boundary value problem for different materials. The forward time centered space is used with the stability conditions. The results were obtained by using MATLAB codes [10].

Hamzeh Zureigat et al. have discussed about an explicit finite difference scheme such as the forward time centered space was implemented to solve the complex fuzzy heat equations. The stability of the proposed approach indicated that the forward time centered space scheme was conditionally stable [11]. This paper proposes the numerical solution for the heat equation using forward time centered space by Python programming. The paper is organized as follows: Section II presents the heat equation, Section III discusses the discrete grid, Section IV focuses on implementation and results, Section V discusses local truncation error, Section VI focuses on stability analysis and finally the conclusion is presented in Section VII.

II. THE HEAT EQUATION

The heat equation is the first order in time (t) and second order in space (x) partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ ....(1)}$$

where $u = u(x,t)$ is the dependent variable. The equation describes heat transfer on a domain

$$\Omega = \{t \geq 0; 0 \leq x \leq 1\} \text{ ....(2)}$$

With an initial condition at time $t = 0$ for all x and boundary condition on the left $x = 0$ and right side $x = 1$.

The forward time centered space difference method for the heat equation with the initial conditions by considering the equation

$$u(x,0) = \begin{cases} 2x; & 0 \leq x \leq \frac{1}{2} \\ 2(1-x); & \frac{1}{2} \leq x \leq 1 \end{cases} \text{ ....(3)}$$

and boundary conditions $u(0,t) = 0$ *and* $u(1,t) = 0$.

III. THE DISCRETE GRID

The region Ω discretized into a uniform mesh Ω_h. The discrete values of x are uniformly spaced in the interval $0 \leq x \leq 1$. In the space x direction into N steps giving a step size of $h = \frac{1-0}{N}$, resulting in $x[i] = 0 + ih$, $i = 0,1,2,\ldots\ldots\ldots,N$ and into N_t steps in the time t direction giving a step size of $k = \frac{1-0}{N_t}$ resulting in $t[j] = 0 + jk$, $j = 0,1,2,\ldots\ldots\ldots,15$.

The Figure 1 below shows the discrete grid points for $N = 10$ *and* $N_t = 100$, the known boundary conditions (Green), initial conditions (Blue) and the unknown values (Red) of the heat equation.

$$N = 10; \quad h = \frac{1}{N} = \frac{1}{10} = 0.1$$

$$N_t = 1000; \quad k = \frac{1}{N_t} = \frac{1}{1000} = 0.001$$

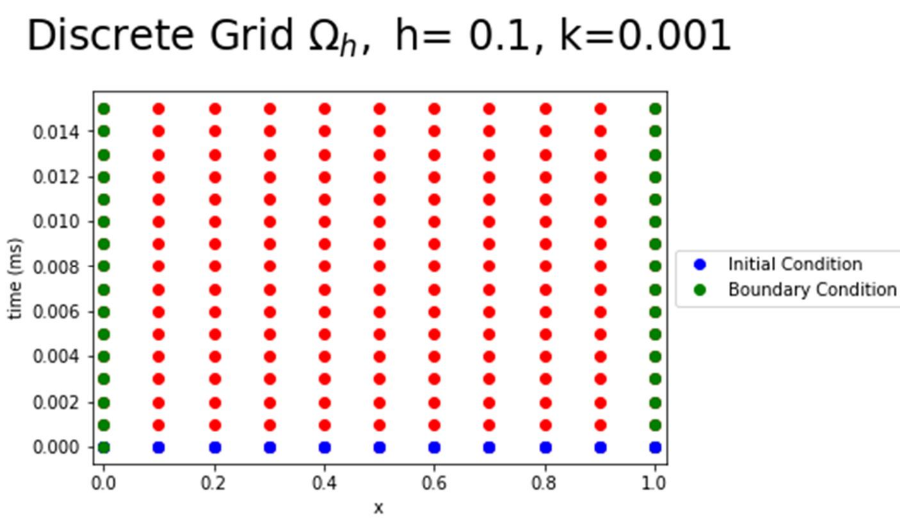

Figure 1.Discrete Grid points for initial and boundary conditions.

Discrete initial and boundary conditions:

The discrete initial conditions are

$$w[i,0] = \begin{cases} 2x[i]; & 0 \leq x[i] \leq \frac{1}{2} \\ 2(1-x[i]); & \frac{1}{2} \leq x[i] \leq 1 \end{cases} \text{ ....(4)}$$

and the discrete boundary conditions $w[0,j] = 0$ *and* $w[10,j] = 0$, where $w[i,j]$ is the numerical approximation of $U(x[i],t[j])$. The Figure 2 below plots the values of $w[i,0]$ for the initial (blue) and boundary (green) conditions for $t[0] = 0$.

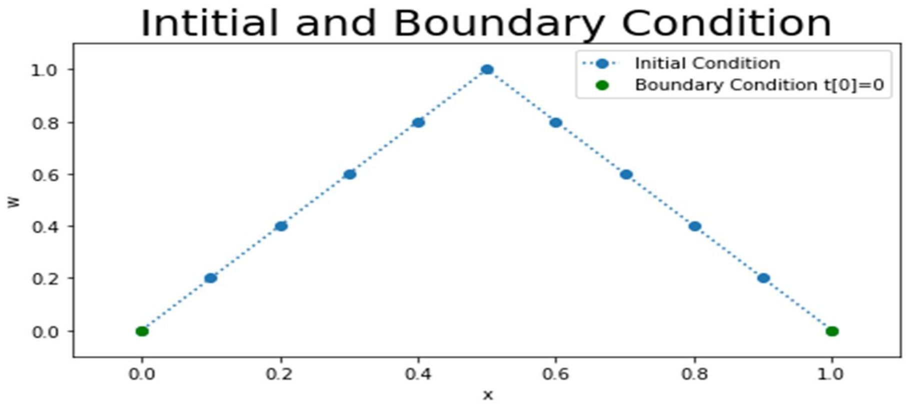

Figure 2.Plot of initial and boundary conditions for $t[0] = 0$.

The explicit forward time centered space difference equation of the heat equation is derived by discretizing around (x_i, t_j)

$$\frac{\partial u_{ij}}{\partial t} = \frac{\partial^2 u_{ij}}{\partial x^2} \ldots\ldots\ldots\ldots (5)$$

The difference equation is given by

$$\frac{w_{ij+1} - w_{ij}}{k} = \frac{w_{i+1j} + w_{i-1j} - 2w_{ij}}{h^2} \ldots\ldots\ldots\ldots\ldots (6)$$

Rearranging the equation (6) we obtain,

$$w_{ij+1} - w_{ij} = \frac{k[w_{i+1j} + w_{i-1j} - 2w_{ij}]}{h^2}$$

$$w_{ij+1} - w_{ij} = r\left[w_{i+1j} + w_{i-1j} - 2w_{ij}\right] \;\; for\ i = 1,2,\ldots\ldots,9\ where\ r = \frac{k}{h^2}$$

$$w_{ij+1} = w_{ij}(1-2r) + rw_{i-1j} + rw_{i+1j} \ldots\ldots\ldots\ldots\ldots (7)$$

This gives the formula for unknown term w_{ij+1} at the $(ij+1)$- mesh points in terms of $x[i]$ along the jth time row. Hence we can calculate the unknown pivotal values of w along the first row of $j = 1$ in terms of the known boundary conditions [12].

This can be written in the equation of matrix form $\quad w_{j+1} = Aw_j + b_j \ldots\ldots\ldots\ldots (8)$

where A is a 9×9 matrix.

$$\begin{bmatrix} w_{1j+1} \\ w_{2j+1} \\ w_{3j+1} \\ w_{4j+1} \\ w_{5j+1} \\ w_{6j+1} \\ w_{7j+1} \\ w_{8j+1} \\ w_{9j+1} \end{bmatrix} = \begin{bmatrix} 1-2r & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ r & 1-2r & r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 1-2r & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1-2r & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 1-2r & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 1-2r & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r & 1-2r & r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & 1-2r & r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & 1-2r \end{bmatrix} \begin{bmatrix} w_{1j} \\ w_{2j} \\ w_{3j} \\ w_{4j} \\ w_{5j} \\ w_{6j} \\ w_{7j} \\ w_{8j} \\ w_{9j} \end{bmatrix} + \begin{bmatrix} rw_{1j} \\ rw_{2j} \\ rw_{3j} \\ rw_{4j} \\ rw_{5j} \\ rw_{6j} \\ rw_{7j} \\ rw_{8j} \\ rw_{9j} \end{bmatrix}$$

It is assumed that the boundary values are known for $j = 1,2,\ldots\ldots and\ for\ i = 0,1,2,\ldots\ldots\ldots$ is the initial condition. The Figure 3 below shows the values of the 9×9 matrix in colour plot form for $r = \frac{k}{h^2}$.

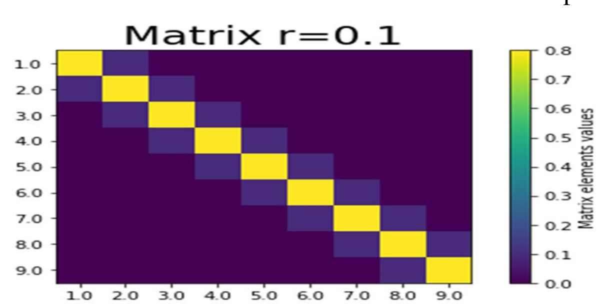

Figure 3.The values of the matrix of order 9×9

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IV. IMPLEMENTATION AND RESULTS

To approximate the solution numerically at $t[1]$, the matrix equation becomes $w_1 = Aw_0 + b_0$ where all the right hand side is known. To approximate the solution numerically at $t[2]$, the matrix equation becomes $w_2 = Aw_1 + b_1$ where all the right hand side is known. To approximate the solution numerically at $t[3]$, the matrix equation becomes $w_3 = Aw_2 + b_2$ where all the right hand side is known. To approximate the solution numerically at $t[4]$, the matrix equation becomes $w_4 = Aw_3 + b_3$ where all the right hand side is known and so on. Each set of numerical solution $w[i,j]$ for all i at the previous time step is used to approximate the solution $w[i,j+1]$. The Figure 4 below shows the numerical approximation $w[i,j]$ of the heat equation using the forward time centered space at

$x[i]\ for\ i = 0,1,2,\ldots\ldots\ldots,10\ \ and\ time\ steps\ t[j]\ for\ j = 1,2,3,\ldots\ldots,15.$

The left plot shows the numerical approximation $w[i,j]$ as a function of $x[i]$ with each colour representing the different time steps $t[j]$.

The right plot shows the numerical approximation $w[i,j]$ as colour plot as a function of $x[i]$ on the X - axis and time $t[j]$ on the Y-axis.

For $r > \frac{1}{2}$, the method is unstable and results a solution that oscillates unnaturally between positive and negative values for each time step. Stable solutions with the forward time centered space scheme are obtained only if $r < \frac{1}{2}$.

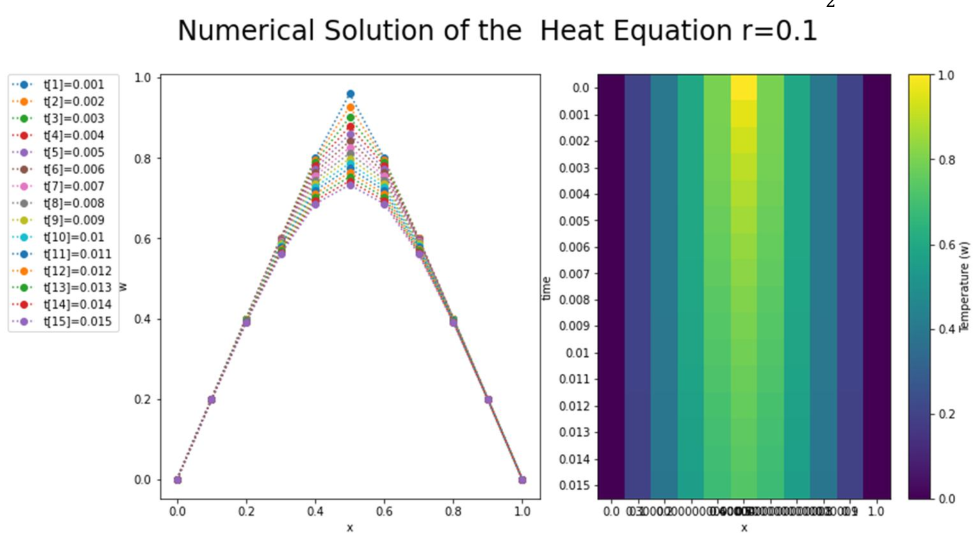


Figure 4.Numerical approximation $w[i,j]$ of the heat equation using the forward time centered space method.

V. TRUNCATION ERROR

The local truncation error of the classical explicit difference approach to the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0\ with$$

$$V_{ij}(w) = \frac{w_{ij+1} - w_{ij}}{k} - \frac{w_{i+1j} + w_{i-1j} - 2w_{ij}}{h^2} = 0$$

$$Let\ T_{ij} = V_{ij}(U) = \frac{U_{ij+1} - U_{ij}}{k} - \frac{U_{i+1j} + U_{i-1j} - 2U_{ij}}{h^2} \ldots\ldots\ldots\ldots \quad (9)$$

By Taylor's expansion, we have

$$U_{i+1j} = U\big((i+1)h, jk\big)$$

$$U_{i+1j} = U\big(x_i + h, t_j\big)$$

$$U_{i+1j} = U_{ij} + h\left(\frac{\partial U}{\partial x}\right)_{ij} + \frac{h^2}{2}\left(\frac{\partial^2 U}{\partial x^2}\right)_{ij} + \frac{h^3}{6}\left(\frac{\partial^3 U}{\partial x^3}\right)_{ij} + \cdots \quad (10)$$

$$U_{i-1j} = U\big((i-1)h, jk\big)$$

$$U_{i-1j} = U\big(x_i - h, t_j\big)$$

$$U_{i-1j} = U_{ij} - h\left(\frac{\partial U}{\partial x}\right)_{ij} + \frac{h^2}{2}\left(\frac{\partial^2 U}{\partial x^2}\right)_{ij} - \frac{h^3}{6}\left(\frac{\partial^3 U}{\partial x^3}\right)_{ij} + \cdots \quad (11)$$

$$U_{ij+1} = U(ih,(j+1)k)$$

$$U_{ij+1} = U(x_i,t_j + k)$$

$$U_{ij+1} = U_{ij} + k\left(\frac{\partial U}{\partial t}\right)_{ij} + \frac{k^2}{2}\left(\frac{\partial^2 U}{\partial t^2}\right)_{ij} + \frac{k^3}{6}\left(\frac{\partial^3 U}{\partial t^3}\right)_{ij} + \cdots \quad (12)$$

Substituting (10), (11) and (12) in (9), we obtain

$$T_{ij} = \left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{ij} + \frac{k}{2}\left(\frac{\partial^2 U}{\partial t^2}\right)_{ij} - \frac{h^2}{12}\left(\frac{\partial^4 U}{\partial x^4}\right)_{ij} + \frac{k^2}{6}\left(\frac{\partial^3 U}{\partial t^3}\right)_{ij} - \frac{h^4}{360}\left(\frac{\partial^6 U}{\partial x^6}\right)_{ij} + \cdots$$

But U is the solution of the differential equation

$$\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{ij} = 0$$

The principal part of local truncation error is

$$\frac{k}{2}\left(\frac{\partial^2 U}{\partial t^2}\right)_{ij} - \frac{h^2}{12}\left(\frac{\partial^4 U}{\partial x^4}\right)_{ij}$$

Hence the truncation error is $T_{ij} = O(k) + O(h^2)$ ....(13)

VI. STABILITY ANALYSIS

To discuss the stability of the explicit forward time centered space difference method of the heat equation, we will use the von Neumann method. The forward time centered space difference equation is

$$\frac{w_{ab+1} - w_{ab}}{k} = \frac{w_{a+1b} + w_{a-1b} - 2w_{ab}}{h^2} \text{............}(14)$$

Approximating $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ at (ah, bk)

Substituting $w_{ab} = \alpha^b e^{i\beta x}$ into the difference equation, we obtain

$$\alpha^{b+1}e^{i\beta ah} - \alpha^b e^{i\beta ah} = r\{\alpha^b e^{i\beta(a-1)h} - 2\alpha^b e^{i\beta ah} + \alpha^b e^{i\beta(a+1)h}\} \; where\; r = \frac{k}{h^2}$$

Dividing by $\alpha^b e^{i\beta ah}$, we obtain $\alpha - 1 = r[e^{i\beta h} + e^{-i\beta h} - 2]$

$$\Rightarrow \alpha = 1 + r[2cos\beta h - 2]$$

$$\alpha = 1 + 2r[cos\beta h - 1]$$

$$\alpha = 1 - 4r\left[sin^2\left(\frac{\beta h}{2}\right)\right]$$

Hence $$\left|1 - 4r\left[sin^2\left(\frac{\beta h}{2}\right)\right]\right| \leq 1$$

$$\Rightarrow 4r\left[sin^2\left(\frac{\beta h}{2}\right)\right] \leq 2$$

$$\Rightarrow r \leq \frac{1}{2}$$

The equation is conditionally stable as $0 < \alpha \leq 1 \; for\; r < \frac{1}{2} \; and\; for\; all\; \beta$.

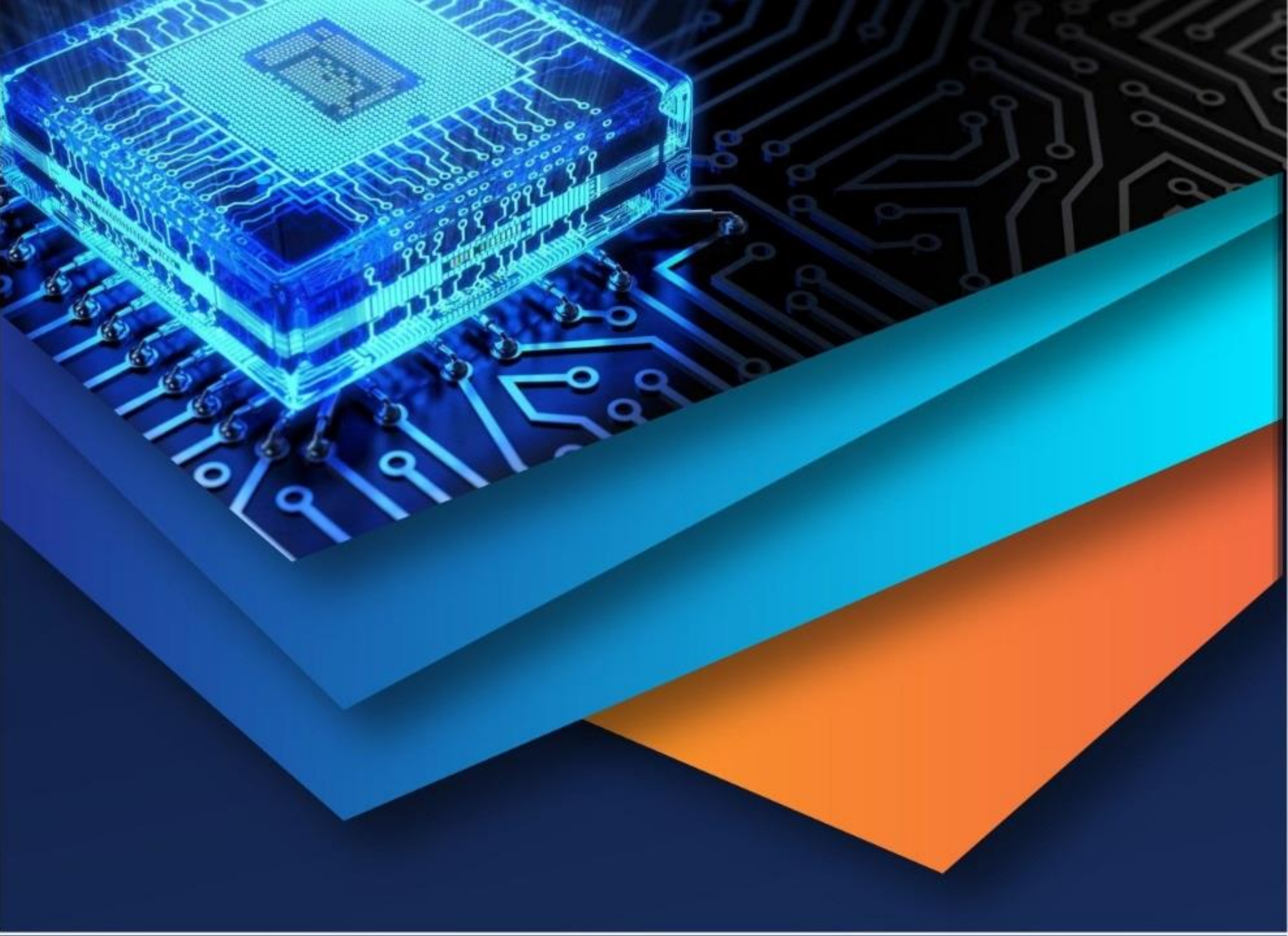
VII. CONCLUSION

We first introduced the one-dimensional heat equation to obtain the numerical approximation using forward time centered space difference method. We have proposed the initial value problem with boundary conditions to find the numerical solution. The region is discretized into a uniform mesh.

The solution is obtained by implementing Python programming by using initial and boundary conditions. The numerical solution for the heat equation is shown in Figure 4. We have investigated the local truncation error and stability analysis of the forward time centered space difference method of the heat equation.

REFERENCES

[1] Xiao-Jun and Feng Gao, "A New Technology for Solving Diffusion and Heat Equation", Thermal Science, Volume 21, January 2016, pp. 01-09.
[2] Alice Gorguis and Wai Kit Benny Chan, "Heat Equation and its Comparative Solutions", Computers and Mathematics with Applications, Elsevier, Volume 55, Issue 12, June 2008,pp. 2973-2980.
[3] Gerald W. Recktenwald, "Finite Difference Approximations to the Heat Equation", Mechanical Engineering, Volume 10(1), January 2004, pp 01-14.
[4] Clint N.Dawson et al., "A Difference Domain Decomposition Algorithm for Numerical Solution for Heat Equation", Mathematics of Computation Published by American Mathematical Society Math. Comp 57 (1991), pp. 63-71.
[5] Abdulla-Al-Mameen et al., "A Study of an Analytical Solution 1D Heat Equation of a Parabolic Partial Differential Equation and implement in Computer Programming", International Journal of Scientific and Engineering Research, Issue 9, 913, pp.01-12.
[6] Jesus Martin-Vaqueno and Svajunas Sajavicious, "The two Level Finite Difference Schemes for the Heat equation with Non-local initial Conditions", Applied Mathematics and Computation, Volume 342, February 2019, pp.166-177.
[7] Hooshmandasl M.R. et al., "Numerical Solution of the One-dimensional Equation by using Chebyshev Wavelets method", Journal of Applied and Computational Mathematics, Volume 1, Issue 6,December 2012, pp.01-07.
[8] Tahirch. N.A.Shahid et al., "Analysis of Stability and Accuracy for Forward Time Centered Space Approximation by Using Modified Equation", Science International (Lahore), 27(3), 2015, pp.2027-2029.
[9] Jeffry Kusuma et al, "On FTCS Approach for Box Model of Three- Dimension Advection-Diffusion Equation", Hindawi International Journal of Differential Equations, Volume 2, November 2018, pp.01-09.
[10] Wahida Zaman Loksar and Rama Sarkar, "A Numerical Solution of Heat Equation for Several Thermal Diffusitivity using Finite Difference Scheme with Stability Conditions", Journal of Applied Mathematics and Physics, Volume 10, 2022, pp.449-465.
[11] Hamzeh Zureigat et al., "The Numerical Solution of Fuzzy Heat Equation with Complex Dirichlet Conditions", International Journal of Fuzzy Logic and Intelligent Systems, Volume 23(1), 2023, pp.11-19.
[12] G D Smith, "Numerical Solution of Partial Differential Equations: Finite Difference Method" Oxford 1992.



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