iJRASET
International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 12 **Issue:** III **Month of publication:** March 2024

DOI: https://doi.org/10.22214/ijraset.2024.58971

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.538
Volume 12 Issue III Mar 2024- Available at www.ijraset.com

On Integer Solutions of the Ternary Bi-Quadratic Diophantine Equation $14(x^2+y^2)-24xy=17z^4$

G.Janaki[1], R.Roshma[2]
[1]Associate Professor, [2]PG Student, PG and Research Department of Mathematics, Cauvery College for Women (Autonomous), Affiliated to Bharathidasan University, Trichy-18.

***Abstract:** This analysis is discussed on the non-trivial distinct integral solution to the ternary bi-quadratic diophantine equation* $14(x^2+y^2)-24xy=17z^4$*, in four dissimilar patterns. Few connections between the solutions are obtained and also unique polygonal numbers are described.*
***Keywords:** Bi-quadratic diophantine equation, integral solutions, polygonal numbers, Star number, Gnomonic number.*

I. INTRODUCTION

Number theory is a vast and attracting field of Mathematics involved with the properties of numbers in general and numbers in unusual as well as the wider classes of problems that proceed from their study. Number theory helps to protect economical data and prevents unofficial use of credit card providing a layer of security for every day transactions and also has connection in telecommunications and data transmission.

Diophantus of Alexandria, an ancient Greek Mathematician who stay around the 3rd century is referred to as the father of polynomials. He made crucial involvement to the study of algebra, particularly in the field of polynomial equations. The quadratic equation, as we notice these days was first debate and educate by Muhammed ibn Musa al-Khwarizmi. At the instruction of his Caliph, he collected all the evidence he could be present on algebra and wrote the initial text on the topic.

A bi-quadratic diophantine equation is a polynomial expression that only uses the zeroth, second, and fourth powers of a variable. Bi-quadratic diophantine equations generally take the form of $ax^4+bx^3+cx^2+dx+e=0$. The main use for bi-quadratic diophantine equations is such that equations with odd degrees cannot be solved using it. One way to transform a polynomial equation into a bi-quadratic diophantine equation is to include it in a bi-quadratic substitution. The assumption approach is the method employed in this conversion.

There are various dissimilar ternary bi-quadratic diophantine equations. To learn the fundamentals of number theory refer[1-3].To understand it in more exhaustive [4-6] is noticeable. For the inherent decrypt to ternary bi-quadratic diophantine equations [7-15] is observable. This article discuss about bi-quadratic diophantine equation $14(x^2+y^2)-24xy=17z^4$ for their non-trivial integral solution patterns. Also connections between the solutions and unique Star, Gnomonic, Centered, Hexagonal numbers are discussed.

A. Definitions

DEFINITION:1.1

A Diophantine equation involving more than one variable in which the co-efficient of the variables are integers and for which the integer solutions are sought.

DEFINITION:1.2

An equation of the form, $ax+by=c$ with $a\neq 0, b\neq 0$ and c are integer is called a linear Diophantine equation in two unknowns x and y.

DEFINITION:1.3

A polynomial equation of degree 2 is said to be Quadratic equation.

Its general form is,

$$ax^2+by+c=0$$

where x is a variable and a,b and c are constants. An equation includes only second degree polynomial.

DEFINITION:1.4

A general quadratic diophantine equation in two variable x and y is given by,

$$ax^2+cy^2=k$$

where a,c and k are positive or negative satisfying the equation.

DEFINITION:1.5

If an integer $m\neq 0$ and $m/a-b$, we say that a is congruent to b modulo m and write,

$$a\equiv b(\mathrm{mod}\, m)$$

DEFINITION:1.6

A Perfect square is a number that can be expressed as k^2, where k is an integer.

Example:

$10^2=100$

DEFINITION:1.7

A Nasty number is a non-negative integer with at least four distinct factors such that the difference between the one pair of factors is equal to the addition of the another pair and the multiplication of each pair is equal to the number.

Example:

6 is a nasty number.

$6=6\times 1=2\times 3$ and $6-1=2+3$

DEFINITION:1.9

A **Pronic number** is a number which is the product of two consecutive integer. If a number 'n' is a product of x and x+1 then n is a pronic number.

Example:

$6=2\times 3$ is a pronic number.

DEFINITION:1.10

A Palindrome number is a number that remains same when its digits are reversed.

Example:

121 is palindrome number.

DEFINITION:1.11

An Harshad number is a number divisible by the sum of its digits.

Example:

20 is harshad number.

DEFINITION:1.12

A Duck number is a number which has zeroes present in it, but there should be no zero present in the beginning of the number.

Example:

3210,7056,8430709 are duck numbers.

DEFINITION:1.13

A Perfect cube is a number that can be expressed as k^3, where k is an integer.

Example:

$10^3=1000$

DEFINITION:1.14

The numbers which can be generated by multiplying the two positive integers and contain at least one divisor other than the number '1' and itself are known as composite numbers.

Example:

4,6,8,9 are composite numbers.

DEFINITION:1.15

A Happy number is a number which eventually reaches 1 when replaced by the sum of the square of each digit.

Example:

13 is a Happy number.

$$13 \to 1^2 + 3^2 = 10$$

$$10 \to 1^2 + 0^2 = 1$$

DEFINITION:1.16

If the sum of square of digits of the numbers does not end with one then the number is said to be sad number.

Example:

4 is a sad number

$$4 \to 4^2 = 16$$

$$16 \to 1^2 + 6^2 = 37$$

$$37 \to 3^2 + 7^2 = 58$$

$$58 \to 5^2 + 8^2 = 83$$

$$83 \to 8^2 + 3^2 = 145$$

$$145 \to 1^2 + 4^2 + 5^2 = 42$$

$$42 \to 4^2 + 2^2 = 20$$

$$20 \to 2^2 + 0^2 = 4$$

B. Notations

$T_{m,n}$ =Polygonal number of rank n with size m

P_n^m =Pyramidal number of rank n with size m

Gno_n =Gnomonic number of rank n

$Star_n$ =Star number of rank n

CH_n =Centered Hexagonal number of rank n

TT_n =Truncated tetrahedral number of rank n

Nex_n =Nexus number of rank n

PT_n =Pentatope number of rank n

TO_n =Truncated octahedral number of rank n

O_n =Octahedral number of rank n

CS_n =Centered square number of rank n

$4DF_n$ =Four dimensional figurate number whose generating Polygon is a square

RD_n =Rhombic dodecagonal number of rank n

CC_n =Centered cube number of rank n

HO_n =Hauy octahedral number of rank n

II. METHOD OF ANALYSIS

In order to find a non-zero distinct integral solution to the ternary bi-quadratic diophantine equation

$$14(x^2+y^2)-24xy=17z^4 \tag{1}$$

On substitution of the linear transformation,

$$\left.\begin{aligned} x &= 2u+2v \\ y &= 2u-2v \\ z &= 2w \end{aligned}\right\} \tag{2}$$

in (1) leads to

$$u^2+13v^2=17w^4 \tag{3}$$

The following list offers four different patterns of solutions for (3). The appropriate values of x and y are derived from the values of u and v by using (2).

PATTERN:I

Assume that

$$w=a^2+13b^2,\ a,b\neq 0 \tag{4}$$

Let $17=(2+i\sqrt{13})(2-i\sqrt{13})$ (5)

Applying the factorization method and substituting (4) and (5) in (3),

$$(u+i\sqrt{13}v)=2a^4-156a^2b^2+338b^4+676ab^3-52a^3b-104iab^3\sqrt{13}+8ia^3b\sqrt{13}+ i\sqrt{13}a^4-78a^2b^2i\sqrt{13}+169b^4i\sqrt{13}$$

The values of u and v are determined by equating the real and imaginary parts as

$$\left.\begin{aligned} u &= 2a^4-156a^2b^2+338b^4+676ab^3-52a^3b \\ v &= -104ab^3+8a^3b+a^4-78a^2b^2+169b^4 \\ w &= a^2+13b^2 \end{aligned}\right\}$$

Substituting the above values of u and v in equation (2), the solution of (1) are

$$x(a,b)=6a^4-468a^2b^2-88a^3b+1144ab^3+1014b^4$$

$$y(a,b)=2a^4-156a^2b^2-120a^3b+1560ab^3+338b^4$$

$$z(a,b)=2a^2+26b^2$$

Properties

1. $x(a,1)-z(a,1)-Nex_a-24PT_a+TO_a+132O_a-6TT_a-CH_a-249CS_a-864Gno_a-a\equiv 0(\bmod 1576)$
2. $x(a,1)-y(a,1)-Nex_a+48DF_a-TO_a-RD_a-CC_a+25T_{12,a}+276Gno_a\equiv 0(\bmod 407)$
3. $y(1,1)-x(1,1)$ is a perfect square
4. $x(1,1)-3z(1,1)$ is harshad number
5. $y(1,1)-8z(1,1)$ is a duck number
6. $y(a,a)-x(a,a)-768DF_a-2T_{18,a}-7Gno_a\equiv 0(\bmod 7)$
7. $x(a,a)-z(a,a)-77184DF_a-158T_{22,a}-83Gno_a-a\equiv 0(\bmod 83)$
8. $y(1,a)-x(1,a)+135Nex_a+24PT_a-5640P_a^3+Star_a-211T_{12,a}-1213Gno_a\equiv 0(\bmod 1345)$

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.538
Volume 12 Issue III Mar 2024- Available at www.ijraset.com

PATTERN:II

Rewrite (3) as,

$$u^2 + 13v^2 = 17w^4 \cdot 1 \tag{6}$$

On the above equation, let us take

$$1 = \frac{(6+i\sqrt{13})(6-i\sqrt{13})}{7} \tag{7}$$

Following the same procedure as in pattern-I, non-zero dissimilar integer solutions to (1) are obtained as

$$x(a,b) = 4802a^4 - 374556a^2b^2 - 288120a^3b + 3745560ab^3 + 811538b^4$$

$$y(a,b) = -6174a^4 + 481578a^2b^2 - 282632a^3b + 3674216ab^3 - 1043406b^4$$

$$z(a,b) = 98a^2 + 1274b^2$$

Properties

1. $y(1,1)$ is a nasty number
2. $2z(1,1)$ is a perfect cube
3. $x(1,1) - y(1,1)$ is a duck number
4. $x(1,1) + y(1,1)$ is a harshad number
5. $x(1,1) - y(1,1) + z(1,1)$ is a happy number
6. $x(1,1) + z(1,1)$ is a sad number
7. $z(a,a) - 98T_{30,a} - 637Gno_a \equiv 0 \pmod{637}$
8. $x(1,1)$ is a composite number

PATTERN:III

Rewrite (3) as,

$$u^2 + 13v^2 = 13w^4 + 4w^4 \tag{8}$$

The equation (8) can be written in the form of

$$\frac{u+2w^2}{w^2+v} = \frac{13(w^2-v)}{u-2w^2} = \frac{\alpha}{\beta}, \beta \neq 0$$

By solving the above system, we get

$$\left.\begin{aligned} u &= 338a^4 + 2b^4 - 156a^2b^2 + 676a^3b - 52ab^3 \\ v &= -169a^4 - b^4 + 78a^2b^2 + 104a^3b - 8ab^3 \end{aligned}\right\} \tag{9}$$

when,

$$w = 13a^2 + b^2$$

$$\alpha = 13a^2 - b^2$$

$$\beta = 2ab$$

Substituting (9) in (2), the solution of equation (1) as

$$x(a,b) = 338a^4 - 156a^2b^2 + 1560a^3b - 120ab^3 + 2b^4$$

$$y(a,b) = 1014a^4 - 468a^2b^2 + 1144a^3b - 88ab^3 + 6b^4$$

$$z(a,b) = 26a^2 + 2b^2$$

Properties

1. $66Nex_a + 384DF_a + 225HO_a + 71T_{18,a} - 876Gno_a - a - x(a,1) - z(a,1) \equiv 0 \bmod (197)$
2. $x(a,a) - z(a,a) - 77952DF_a - 532CH_a - 798Gno_a \equiv 0 (\bmod 266)$
3. $2x(1,1) - 2z(1,1)$ is pronic number
4. $z(1,1)$ is a happy number
5. $y(1,1)$ is duck number
6. $x(a,a) - y(a,a) - 768DF_a - 2T_{18,a} - 7Gno_a \equiv 0 (\bmod 7)$
7. $y(1,a) - z(1,a) - 24PT_a - Nex_a + 26RD_a + 1294T_{3,a} - 442Gno_a \equiv 0 (\bmod 1358)$
8. $x(1,a) - y(1,a) + Nex_a - 48DF_a - 33O_a + CH_a - 163CS_a - 366Gno_a - a \equiv 0 (\bmod 877)$

PATTERN: IV

The equation (8) can be written in the form as

$$\frac{u+2w^2}{13(w^2+v)} = \frac{w^2-v}{u-2w^2} = \frac{\alpha}{\beta}, \beta \neq 0$$

By solving the above system, we get

$$\left.\begin{aligned} u &= -338a^4 - 2b^4 + 156a^2b^2 + 676a^3b - 52ab^3 \\ v &= 169a^4 + b^4 - 78a^2b^2 + 104a^3b - 8ab^3 \end{aligned}\right\} \quad (10)$$

when,

$$w = 13a^2 + b^2$$

$$\alpha = 2ab$$

$$\beta = 13a^2 - b^2$$

Substituting (10) in (2), the solution of equation (1) as

$$x(a,b) = -338a^4 + 156a^2b^2 + 1560a^3b - 120ab^3 - 2b^4$$

$$y(a,b) = -1014a^4 + 468a^2b^2 + 1144a^3b - 88ab^3 - 6b^4$$

$$z(a,b) = 26a^2 + 2b^2$$

Properties

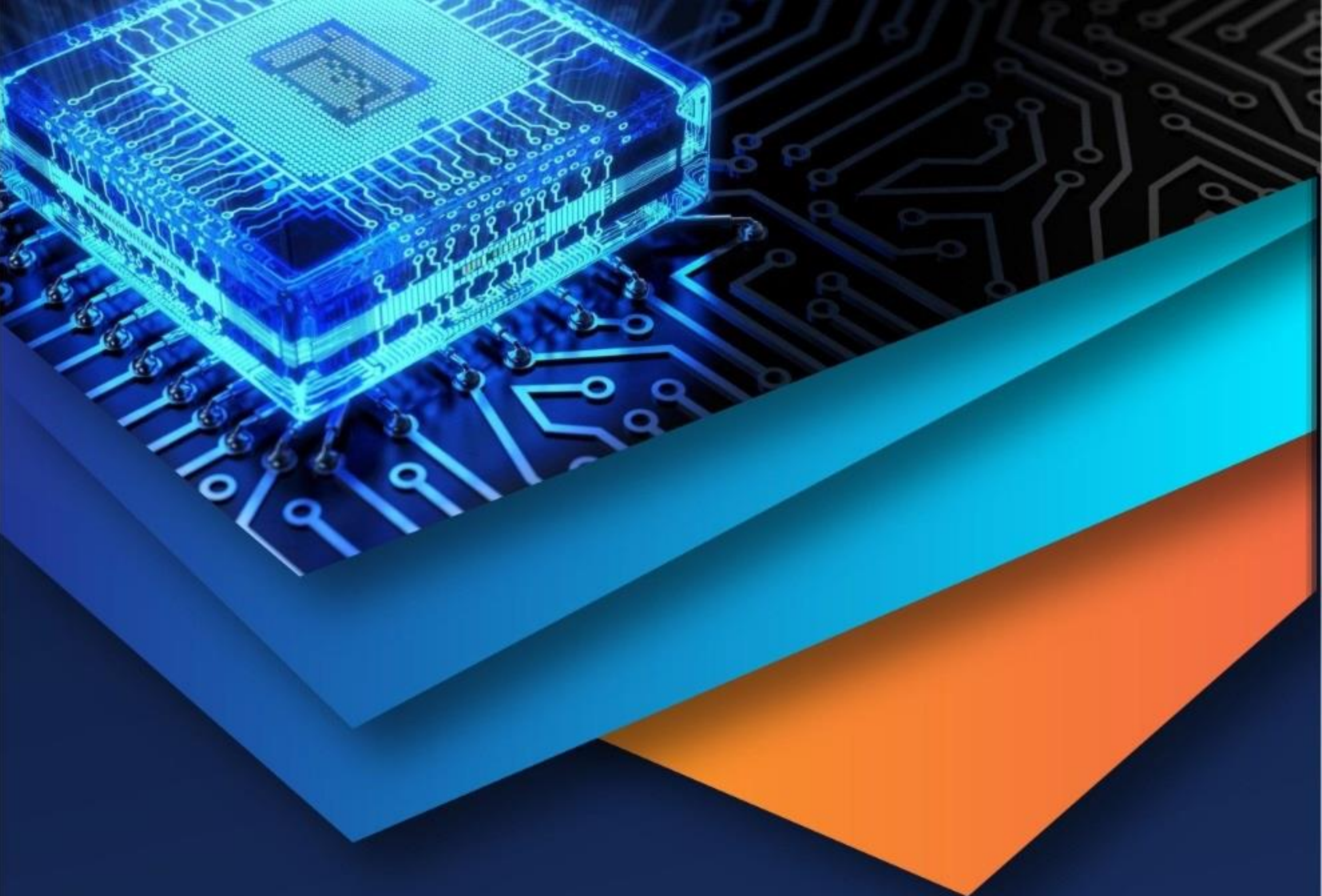
1. $x(a,a) - y(a,a) - 36096DF_a + 8Star_a + 24Gno_a \equiv 0 (\bmod 16)$
2. $x(1,a) - y(1,a) - Nex_a + 48DF_a + 63O_a + 34T_{21,a} + 630Gno_a + a \equiv 0 (\bmod 1305)$
3. $x(1,2) - y(1,1) + z(1,1)$ is harshad number
4. $z(1,1)$ is happy number
5. $y(1,1) - x(1,1) - z(2,2)$ is palindrome number
6. $z(a,a) - 14CS_a - 14Gno_a = 0$
7. $z(a,1) - x(a,1) - 8112PT_a + 546HO_a + 234T_{30,a} + 411Gno_a \equiv 0 (\bmod 1231)$
8. $y(a,a) + z(a,a) - 24192DF_a - 238CS_a - 238Gno_a = 0$

III. CONCLUTION

In this analysis, the ternary bi-quadratic diophantine equation $14(x^2+y^2)-24xy=17z^4$ for its non-zero distinct integral solutions in four different patterns are explored and also few interesting properties are determined. To conclude, one may search for solving the ternary bi-quadratic problem under consideration as well as bi-quadratic diophantine equations with many variables.

REFERENCES

[1] Carmichael, R.D., The theory of numbers and Diophantine Analysis, Dover Publications, New York, 1959.
[2] Dickson L.E, History of Theory of Numbers, Vol.11, Chelsea Publishing company, New York,1952.
[3] Mordell. L.J, Diophantine equations, Academic Press, London,
[4] Nigel D.Smart, The Algorithmic Resolutions of Diophantine Equations, Cambridge University Press, London, 1999.
[5] Janaki, Dr G. "Explorations in Diophantine Equations.", B P International , 2023.
[6] S.G. Telang, Number Theory, Tata Mc Graw Hill publishing company, New Delhi, 1996.
[7] Gopalan M.A and Janaki G, "Integral solutions of the ternary quartic equation $x^2-y^2+xy=z^4$ ",Impact J.Sci. Tech, Vol 2, issue 2,71-79,2008.
[8] Gopalan M.A and V.Geetha, "Integral solutions of ternary biquadratic equation $x^2+13y^2=z^4$ "International Journal of Latest Research in Science and Technology ISSN, Volume 2, Issue 2 (2013).
[9] Gopalan M.A and V.Geetha, "On The Non-Homegeneous Biquadratic Equation with Four Unknowns $x^3+y^3+z^3=128(x+y)w^3$ ", Scholars Journal of Physics, Mathematics and Statistics,Vol-1, Issue-2, 2014.
[10] Janaki G, S. Vidhya, "On the Integer Solutions of the Homogeneous Biquadratic Diophantine Equation $x^4-y^4=82(z^2-w^2)p^2$ ",International Journal of Engineering Science and Computing, June 2016.
[11] Janaki G, C Saranya, "Integral Solutions of the Homogeneous Biquadratic Diophantine Equation $3(x^4-y^4)-2xy(x^2-y^2)=972(z+w)p^3$ ",International Journal for Research in Applied Science & Engineering Technology (IJRASET), 2017.
[12] Janaki G, Gowri Shankari A, "Integral Solution of the Binary Quadratic Equation $3x^2-y^2=2$ that depicts parabolas and hyperbolas. Volume 10, Issue.1, pp.10-13, February 2023, E-ISSN: 2348-4519.
[13] P. Saranya , K.Poorani, " Intrinsic Solutions of Diophantine Equation Involving centered Square number $E^4-H^4=(n^2+(n-1)^2)(k-l)R^2$ ", International Journal for Research in Applied Science & Engineering Technology(IJRASET), Volume 11, Issue III mar 2023.
[14] Janaki G, Krishnaveni M, "On Integer Solutions of the Homogeneous Biquadratic Diophantine Equation $x^4-y^4=26(z^2-w^2)p^3$," International Journal for Research in Applied Science & Engineering Technology (IJRASET), Volume 11, Issue III, March 2023.
[15] Vidhyalakshmi, S., and M. A. Gopalan. "On Finding Integer Solutions to Non-homogeneous Ternary Bi-quadratic Equation $3(x^2+y^2)-2xy=11z^4$ ". (2022).



10.22214/IJRASET

INDEX COPERNICUS 45.98

IMPACT FACTOR: 7.129

IMPACT FACTOR: 7.429

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY