# On the Integer Solutions of the Homogeneous Biquadratic Diophantine Equation 

$x^{4}-y^{4}=26\left(z^{2}-w^{2}\right) p^{2}$<br>G. Janaki ${ }^{1}$, M. Krishnaveni ${ }^{2}$<br>${ }^{1}$ Associate Professor, ${ }^{2}$ PG Student, Department of Mathematics Cauvery College for Women (Autonomous), Trichy, Tamil Nadu, India


#### Abstract

The homogenous biquadratic Diophantine equation with five unknowns non-zero unique integer solutions $x^{4}-y^{4}=26\left(z^{2}-w^{2}\right) p^{2}$ are found using several techniques. The unusual numbers and the solutions are found to have a


few intriguing relationships. The relationships between the solutions' recurrences are also shown.
Keywords: Five unknowns in a homogenous biquadratic equation, Numbers in polygons.

## Notations

$T_{12, n}=$ Dodecagonal number of rank n .
$T_{18, n}=$ Octadecagonal number of rank n .
$T_{22, n}=$ Icosidigonal number of rank n .
$T_{26, n}=$ Icosihexagonal number of rank n .
$T_{28, n}=$ Icosioctagonal number of rank n .
$G n o_{n}=$ Gnomonic number of rank n.

## I. INTRODUCTION

The biquadratic equation can be used to represent the quartic equation. These issues can be resolved using the quadratic formula since they can be reduced to quadratic equations [1-5], they are simple to solve. Several mathematicians have developed an interest in biquadratic Diophantine equations, both homogeneous and non-homogeneous. One can refer to [6-11] in the context for a variety of issues involving the two, three, and four variable Diophantine equations.
This communication examines the non-zero unique integer solutions to the biquadratic equation with five unknowns provided by $x^{4}-y^{4}=26\left(z^{2}-w^{2}\right) p^{2}$.Also the recurrence linkages between the solutions are discovered.

## II. METHOD OF ANALYSIS

In order to get the non-zero unique integral solution to the homogeneous biquadratic Diophantine problem with five unknowns, $x^{4}-y^{4}=26\left(z^{2}-w^{2}\right) p^{2}$
An explanation of the linear transformation

$$
\begin{equation*}
x=u+v, y=u-v, z=2 u+v, w=2 u-v \tag{2}
\end{equation*}
$$

Equation (1) is changed to
$u^{2}+v^{2}=26 p^{2}$

## A. Pattern I

Assume
$26=(5+i)(5-i)$
and $p=a^{2}+b^{2}=(a+i b)(a-i b)$
Using (4) and (5) in (3) and employing the method of factorization, we get,
$(u+i v)(u-i v)=(5+i)(5-i)(a+i b)^{2}(a-i b)^{2}$
Equating the like factors, we get,
$(u+i v)=(5+i)(a+i b)^{2}$
$(u-i v)=(5-i)(a-i b)^{2}$
Equating real and imaginary parts, we get,
$u=5 a^{2}-5 b^{2}-2 a b$
$v=a^{2}-b^{2}+10 a b$
Equation (2) non-zero unique integer solutions are as follows when u and v are substituted:
$x=x(a, b)=6 a^{2}-6 b^{2}+8 a b$
$y=y(a, b)=4 a^{2}-4 b^{2}-12 a b$
$z=z(a, b)=11 a^{2}-11 b^{2}+6 a b$
$w=w(a, b)=9 a^{2}-9 b^{2}-14 a b$
$p=p(a, b)=a^{2}+b^{2}$
Properties

1) $2\left[x(a, 1)+y(a, 1)-T_{22, a}\right]-5 G n o_{a} \equiv 0(\bmod 15)$.
2) $x(1,1)-y(1,1)+z(1,1)-w(1,1)=40$ is Duck number.
3) $2\left[w(a, 1)-p(a, 1)-T_{18, a}\right]+5 G n o_{a} \equiv 0(\bmod 15)$.
4) $z(A, A)=6 A^{2}$ is Nasty number.
5) $2\left[z(a, 1)+p(a, 1)-T_{126, a}\right]+5 G n o_{a} \equiv 0(\bmod 15)$.
B. Pattern II

26 can also be expressed as,
$26=(1+i 5)(1-i 5)$
Applying (5) and (6) in (3) and the factorization approach, we obtain,
$(u+i v)(u-i v)=(1+i 5)(1-i 5)(a+i b)^{2}(a-i b)^{2}$
Similar to pattern 1, the non- zero distinct integer answers of (1) are

$$
\begin{aligned}
& x=x(a, b)=6 a^{2}-6 b^{2}-8 a b \\
& y=y(a, b)=-4 a^{2}+4 b^{2}-12 a b \\
& z=z(a, b)=7 a^{2}-7 b^{2}-18 a b \\
& w=w(a, b)=-3 a^{2}+3 b^{2}-22 a b \\
& p=p(a, b)=a^{2}+b^{2}
\end{aligned}
$$

Properties:

1) $2\left[x(a, 1)+z(a, 1)+w(a, 1)-T_{22, a}\right]-5 G n o_{a} \equiv 0(\bmod 15)$
2) $x(a, 1)+z(a, 1)-T_{28, a}+7$ Gno $_{a}+20=0$
3) $2\left[z(a, 1)-w(a, 1)-T_{22, a}\right]-5 G n o_{a} \equiv 0(\bmod 15)$
4) $y(A, A)-z(A, A)=6 A^{2}$ is Nasty number.
5) $11 \times(-w(1,1))=242$ is Palindrom number.
C. Pattern III

Rewriting equation (3) as
$1 * u^{2}=26 p^{2}-v^{2}$
Assume
$u=26 a^{2}-b^{2}=(\sqrt{26} a+b)(\sqrt{26} a-b)$
Write 1 as,
$1=(\sqrt{26}+5)(\sqrt{26}-5)$
Using (8) and (9) in (7) and employing the method of factorization, we get,

$$
\begin{equation*}
(\sqrt{26}+5)(\sqrt{26}-5)(\sqrt{26} a+b)^{2}(\sqrt{26} a-b)^{2}=(\sqrt{26} p+v)(\sqrt{26} p-v) \tag{9}
\end{equation*}
$$

Equating the like factors, we get,

$$
\begin{aligned}
& (\sqrt{26}+5)(\sqrt{26} a+b)^{2}=(\sqrt{26} p+v) \\
& (\sqrt{26}-5)(\sqrt{26} a-b)^{2}=(\sqrt{26} p-v)
\end{aligned}
$$

Equating rational and irrational parts, we get,

$$
\begin{aligned}
& p=26 a^{2}+b^{2}+10 a b \\
& v=130 a^{2}+5 b^{2}+52 a b
\end{aligned}
$$

Equation (2) non-zero unique integer solutions are as follows when u and v are substituted:

$$
\begin{aligned}
& x=x(a, b)=156 a^{2}+4 b^{2}+52 a b \\
& y=y(a, b)=-10 a^{2}-6 b^{2}-52 a b \\
& z=z(a, b)=182 a^{2}+3 b^{2}+52 a b \\
& w=w(a, b)=-78 a^{2}-7 b^{2}-52 a b \\
& p=26 a^{2}+b^{2}+10 a b
\end{aligned}
$$

Properties:

1) $x(1, b)+z(1, b)+p(1, b)+y(1, b) 2 T_{4, b}-31 G n o_{a} \equiv 0(\bmod 291)$
2) $x(1,1)=212$ is palindrom number.
3) $x(1,1)+y(1,1)=50$ is Duck number.
4) $w(1,1)-y(1,1)=25$ is Square number.
5) For all values of a and b, $x+y$ is divisible by 2
D. Pattern IV

Rewriting equation (3) as,
$1 * v^{2}=26 p^{2}-u^{2}$
Write 1 as,

$$
1=\frac{(\sqrt{26}+1)(\sqrt{26}-1)}{25}
$$

Assume

$$
\begin{equation*}
v^{2}=26 a^{2}-b^{2}=(\sqrt{26} a+b)(\sqrt{26} a+b) \tag{12}
\end{equation*}
$$

Applying (11) and (12) in (10) and the factorization approach, we obtain,

$$
\frac{(\sqrt{26}+1)(\sqrt{26}-1)}{25}(\sqrt{26} a+b)^{2}(\sqrt{26} a-b)^{2}=(\sqrt{26} p+v)(\sqrt{26} p-v)
$$

Equating the like factorization, we get,

$$
\begin{aligned}
& \frac{(\sqrt{26}+1)}{5}(\sqrt{26} a+b)^{2}=(\sqrt{26} p+v) \\
& \frac{(\sqrt{26}-1)}{5}(\sqrt{26} a-b)^{2}=(\sqrt{26} p-v)
\end{aligned}
$$

By equating the rational and irrational components, we obtain,

$$
\begin{align*}
& p=\frac{1}{5}\left(26 a^{2}+b^{2}+2 a b\right)  \tag{13}\\
& u=\frac{1}{5}\left(26 a^{2}+b^{2}+52 a b\right) \tag{13}
\end{align*}
$$

In order to discover only integer solutions, we must substitute $a=5 A$ and $b=5 B$ in equations (12) and (13).

$$
\begin{aligned}
& u=130 A^{2}+5 B^{2}+260 A B \\
& v=650 A^{2}-25 B^{2}
\end{aligned}
$$

Equation (2) non-zero unique integer solutions are as follows when $u$ and $v$ are substituted:

$$
\begin{aligned}
& x=x(A, B)=780 A^{2}-20 B^{2}+260 A B \\
& \quad y=y(A, B)=-520 A^{2}+30 B^{2}+260 A B \\
& z=z(A, B)=910 A^{2}-15 B^{2}+520 A B \\
& w=w(A, B)=-390 A^{2}+35 B^{2}+520 A B \\
& p=130 A^{2}+5 B^{2}+10 A B
\end{aligned}
$$

Properties:

1) $x(1,1)+y(1,1)=790$ is Duck number.
2) $p(1,1)-1=144$ is Square number.
3) $p(1, a)-w(1, a)-x(1, a)+2 T_{12, a}+389 G n o_{a} \equiv 0(\bmod 649)$
4) $x(1,1)-w(1,1)+y(1,1)=625$ is Square number.
E. Pattern V

Equation (3) can be written as,

$$
\begin{align*}
& u^{2}-p^{2}=25 p^{2}-v^{2} \\
& (u+p)(u-p)=(5 p+v)(5 p-v) \tag{14}
\end{align*}
$$

Which is represented in the form of ratio as,

$$
\frac{u+p}{5 p+v}=\frac{5 p-v}{u-p}=\frac{A}{B}, B \neq 0
$$

This is equivalent to the following two equations

$$
\begin{gathered}
B u+(B-5 A) p-A v=0 \\
-A u+(A+5 B) p-B v=0
\end{gathered}
$$

Solving the above equation by cross ratio method, we get,

$$
\begin{aligned}
& u=A^{2}-B^{2}+10 A B \\
& v=-5 A^{2}+5 B^{2}+2 A B \\
& p=A^{2}+B^{2}
\end{aligned}
$$

Substituting u and v in equation (2), the non - zero distinct integer solutions are
$x=x(A, B)=-4 A^{2}+4 B^{2}+12 A B$
$y=y(A, B)=6 A^{2}-6 B^{2}+8 A B$
$z=z(A, B)=-3 A^{2}+3 B^{2}+22 A B$
$w=w(A, B)=7 A^{2}-7 B^{2}+18 A B$
$p=A^{2}+B^{2}$
Properties

1) $2\left[y(a, 1)+w(a, 1)-T_{28, a}\right]-11 G n o_{a} \equiv 0(\bmod 15)$
2) $p(1,1)+w(1,1)=20$ is Duck number
3) For all values of A and B, $x+y$ is divisible by 2 .
4) $2\left[y(a, 1)+x(a, 1)-T_{22, a}\right]-5 G n o_{a} \equiv 0(\bmod 15)$
5) $x(1,1)+y(1,1)+z(1,1)+w(1,1)+2 p(1,1)=64$ is Cube number.

## III. CONCLUSION

Other non-zero unique integer solutions to the multivariable biquadratic equations under consideration may be sought after.

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