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On The Ternary Quadratic Diophantine Equation

$$x^2 + 14xy + y^2 = z^2$$

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Abstract: The non-zero unique integer solutions to the quadratic Diophantine equation with three unknowns $x^2 + 14xy + y^2 = z^2$ are examined. We derive integral solutions in four different patterns. A few intriguing relationships between the answers and a few unique polygonal integers are shown.

Keywords: Ternary quadratic equation, integral solutions.

I. INTRODUCTION

There is a wide range of ternary quadratic equations. One might refer to [1-8] for a thorough review of numerous issues. These findings inspired us to look for an endless number of non-zero integral solutions to another intriguing ternary quadratic problem provided by $x^2 + 14xy + y^2 = z^2$ illustrating a cone for figuring out its many non-zero integral points. A few intriguing connections between the solutions are displayed.

II. CONNECTED WORK

Pr_a = Pronic number of the rank 'n'

Gno_a = Gnomonic number of rank 'n'

$T_{m,n}$ = Polygonal number of rank 'n' with sides 'm'

III. METHODOLOGY

The ternary quadratic Diophantine equation to be solved for its non-zero integral solutions is,

$$x^2 + 14xy + y^2 = z^2 \quad (1) \text{ Replacement of}$$

linear transformations

$$x = \alpha + \beta \text{ and } y = \alpha - \beta \quad (2)$$

(1) results in

$$16\alpha^2 - 12\beta^2 = z^2 \quad (3)$$

We present below different patterns of solving (3) and thus obtain different choices of integer solutions of (1)

A. Pattern: 1

$$\text{Assume, } z = z(a, b) = 16a^2 - 12b^2 \quad (4)$$

Where a and b are non-zero integers.

Substitute (4) in (3) we get,

$$(4\alpha + \sqrt{12}\beta^2)(4\alpha - \sqrt{12}\beta^2) = (4a + \sqrt{12}b)^2(4a - \sqrt{12}b)^2 \quad (5)$$

Equating rational and irrational terms we get,

$$\alpha = \frac{1}{4}[16a^2 + 12b^2]$$

$$\beta = 8ab$$

Replacing the preceding variables of α and β in equation (2), the equivalent answers to (1) are provided by

$$x = x(a, b) = 4a^2 + 3b^2 + 8ab$$

$$y = y(a, b) = 4a^2 + 3b^2 - 8ab$$

$$z = z(a, b) = 16a^2 - 12b^2$$

Now put $a = 4A$ and $b = 4B$ we get,

$$x = 64A^2 + 48B^2 + 128AB$$

$$y = 64A^2 + 48B^2 - 128AB$$

$$z = 256A^2 - 192B^2$$

OBSERVATION:

- 1) $y(A, 2) - x(A, 2) + 256Gno_A \equiv 0 \pmod{256}$
- 2) $2[x(1, 1) + y(1, 1) + z(1, 1)]$ is a Perfect square.
- 3) $x(A, 2) + y(A, 2) + z(A, 2) - 32T_{26,A} - 176Gno_A \equiv 0 \pmod{592}$
- 4) $z(2a, a) - x(3a, 2a) + 88T_{18,a} + 308Gno_a \equiv 0 \pmod{308}$
- 5) $2y(A, 1) - x(A, 1) - 8T_{18,A} + 164Gno_A \equiv 0 \pmod{116}$
- 6) $x(A, A+1) - z(A+1, A) - 176T_{14,A} - 144Gno_A \equiv 0 \pmod{352}$

B. Pattern: 2

Equation (3) can be written as

$$16\alpha^2 - 12\beta^2 = z^2 \times 1 \quad (8)$$

Write 1 as

$$1 = (7 + 2\sqrt{12})(7 - 2\sqrt{12}) \quad (9)$$

By using equation (9) and the value of z , we can write

$(4\alpha + \sqrt{12}\beta)(4\alpha - \sqrt{12}\beta) = (4a + \sqrt{12}b)^2(4a - \sqrt{12}b)^2(7 + 2\sqrt{12})(7 - 2\sqrt{12})$ Equating positive and negative terms we get,

$$\alpha = \alpha(a, b) = 28a^2 + 21b^2 + 48ab$$

$$\beta = \beta(a, b) = 32a^2 + 24b^2 + 56ab$$

Replacing the preceding variables of α and β in equation (2), the equivalent integer answers to (1) are provided by

$$x = x(a, b) = 60a^2 + 45b^2 + 104ab$$

$$y = y(a, b) = -4a^2 - 3b^2 - 8ab$$

$$z = z(a, b) = 16a^2 - 12b^2$$

OBSERVATION:

- 1) $y(a, 1) - x(a, 1) + 16T_{10,a} + 80Gno_a \equiv 0 \pmod{128}$
- 2) $z(a, 1) - y(a, 1) - x(a, 1) + 40T_{4,a} + 48Gno_a \equiv 0 \pmod{102}$

- 3) $9z(1,1) + 2y(1,1)$ is a nasty number
- 4) $10z(a,a) - 4y(a,a)$ is a Perfect square
- 5) $3x(a,a) + 4y(a,a) - 567T_{4,a} = 0$
- 6) $x(a,a) - y(a,a) - 2z(a,a)$ is a cubical integer

C. Pattern: 3

Equation (3) can be written as

$$z^2 + 12\beta^2 = 16\alpha^2 \quad (10)$$

$$\text{Assume, } \alpha = a^2 + 12b^2 \quad (11)$$

$$16 = \frac{(4 + i2\sqrt{12})(4 - i2\sqrt{12})}{4} \quad (12)$$

In equations (11) and (12) in (10) we get,

$$z^2 + 12\beta^2 = \frac{(4 + i2\sqrt{12})(4 - i2\sqrt{12})}{4}(a^2 + 12b^2)^2 \quad (13)$$

Equation (13) as,

$$(z + i\sqrt{12}\beta)(z - i\sqrt{12}\beta) = \left(\frac{4 + i2\sqrt{12}}{2}\right)\left(\frac{4 - i2\sqrt{12}}{2}\right)(a + i\sqrt{12}b)^2(a - i\sqrt{12}b)^2$$

Equating positive and negative terms we get,

$$z = z(a,b) = 2a^2 - 24b^2 - 24ab$$

$$\beta = \beta(a,b) = a^2 - 12b^2 + 4ab$$

Replacing the preceding variables of z and β in equation (2), the equivalent answers to (1) are provided by

$$x = x(a,b) = 2a^2 + 4ab$$

$$y = y(a,b) = 24b^2 - 4ab$$

$$z = z(a,b) = 2a^2 - 24b^2 - 24ab$$

OBSERVATION:

- 1) $x(a,1) - y(a,1) + z(a,1) - 4Pr_a + 8Gno_a \equiv 0(\text{mod } 60)$
- 2) $2y(1,a) - z(1,a) - 48Pr_a - 10Gno_a \equiv 0(\text{mod } 40)$
- 3) $x(2a,a) - y(2a,a) + 88T_{4,a} = 0$
- 4) $x(1,1) - 2z(1,1)$ is a Perfect square
- 5) $x(a,a) - y(a,a) + 14T_{4,a} = 0$
- 6) $x(1,a) + z(a,1) - 2T_{4,a} + 10Gno_a \equiv 0(\text{mod } 32)$

D. Pattern: 4

$$\text{Equation (3) as, } 16\alpha^2 - z^2 = 12\beta^2 \quad (14)$$

Write equation (14) as

$$(4\alpha + z)(4\alpha - z) = 12\beta \times \beta \quad (15)$$

Equation (15) is written in the form of a ratio as

$$\frac{4\alpha + z}{\beta} = \frac{12\beta}{4\alpha - z} = \frac{p}{q}$$

This is equivalent to the following two equations

$$4\alpha q + zq - \beta p = 0 \quad (16)$$

$$4\alpha p - zp - 12\beta q = 0 \quad (17)$$

By using cross-multiplication, we get

$$\alpha = \alpha(a, b) = p^2 + 12q^2$$

$$\beta = \beta(a, b) = 8pq$$

$$z = z(a, b) = 4p^2 - 48q^2$$

Replacing the preceding variables of α and β in equation (2), the equivalent integer answers to (1) are provided by

$$x = x(p, q) = p^2 + 12q^2 + 8pq$$

$$y = y(p, q) = p^2 + 12q^2 - 8pq$$

$$z = z(p, q) = 4p^2 - 48q^2$$

OBSERVATION:

- 1) $x(p, p) + y(p, p) + z(p, p)$ is a Perfect square
- 2) $2z(1, 1) - 16y(1, 1)$ is a cubical integer
- 3) $x(p, p) + y(p, p) + z(p, p) - 6Pr_p \equiv 0 \pmod{30}$
- 4) $x(p, 1) - y(p, 1) + z(p, 1) - 2T_{6,p} - 9Gno_p \equiv 0 \pmod{39}$
- 5) $x(p, p) + z(p, p) + 23p^2 = 0$
- 6) $x(2, p) - z(2, p) - 60T_{4,p} - 8Gno_p \equiv 0 \pmod{4}$

IV. CONCLUSION

In this paper, We have found an endless number of non-zero distinct integer solutions to the ternary quadratic Diophantine equation

$$2(x^2 + y^2) - 3xy = 16z^2$$

To sum up, one can look for other solution patterns and their accompanying attributes.

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