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On The Ternary Quadratic Diophantine Equation $x^2 + 14xy + y^2 = z^2$

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***Abstract:** The non-zero unique integer solutions to the quadratic Diophantine equation with three unknowns $x^2 + 14xy + y^2 = z^2$ are examined. We derive integral solutions in four different patterns. A few intriguing relationships between the answers and a few unique polygonal integers are shown.*
***Keywords:** Ternary quadratic equation, integral solutions.*

I. INTRODUCTION

There is a wide range of ternary quadratic equations. One might refer to [1-8] for a thorough review of numerous issues. These findings inspired us to look for an endless number of non-zero integral solutions to another intriguing ternary quadratic problem provided by $x^2 + 14xy + y^2 = z^2$ illustrating a cone for figuring out its many non-zero integral points. A few intriguing connections between the solutions are displayed.

II. CONNECTED WORK

$\Pr_a =$ Pronic number of the rank 'n'

$Gno_a =$ Gnomonic number of rank 'n'

$T_{m,n} =$ Polygonal number of rank 'n' with sides 'm'

III. METHODOLOGY

The ternary quadratic Diophantine equation to be solved for its non-zero integral solutions is,

$x^2 + 14xy + y^2 = z^2$ (1) Replacement of linear transformations

$x = \alpha + \beta$ and $y = \alpha - \beta$ (2)

(1) results in

$16\alpha^2 - 12\beta^2 = z^2$ (3)

We present below different patterns of solving (3) and thus obtain different choices of integer solutions of (1)

A. Pattern: 1

Assume, $z = z(a,b) = 16a^2 - 12b^2$ (4)

Where a and b are non-zero integers.

Substitute (4) in (3) we get,

$(4\alpha + \sqrt{12}\beta^2)(4\alpha - \sqrt{12}\beta^2) = (4a + \sqrt{12}b)^2(4a - \sqrt{12}b)^2$ (5)

Equating rational and irrational terms we get,

$$\alpha = \frac{1}{4}[16a^2 + 12b^2]$$

$$\beta = 8ab$$

Replacing the preceding variables of α and β in equation (2), the equivalent answers to (1) are provided by

$$x = x(a,b) = 4a^2 + 3b^2 + 8ab$$

$$y = y(a,b) = 4a^2 + 3b^2 - 8ab$$

$$z = z(a,b) = 16a^2 - 12b^2$$

Now put $a = 4A$ and $b = 4B$ we get,

$$x = 64A^2 + 48B^2 + 128AB$$

$$y = 64A^2 + 48B^2 - 128AB$$

$$z = 256A^2 - 192B^2$$

OBSERVATION:

1) $y(A,2) - x(A,2) + 256Gno_A \equiv 0(\bmod 256)$
2) $2[x(1,1) + y(1,1) + z(1,1)]$ is a Perfect square.
3) $x(A,2) + y(A,2) + z(A,2) - 32T_{26,A} - 176Gno_A \equiv 0(\bmod 592)$
4) $z(2a,a) - x(3a,2a) + 88T_{18,a} + 308Gno_a \equiv 0(\bmod 308)$
5) $2y(A,1) - x(A,1) - 8T_{18,A} + 164Gno_A \equiv 0(\bmod 116)$
6) $x(A,A+1) - z(A+1,A) - 176T_{14,A} - 144Gno_A \equiv 0(\bmod 352)$

B. Pattern: 2

Equation (3) can be written as

$$16\alpha^2 - 12\beta^2 = z^2 \times 1 \quad (8)$$

Write 1 as

$$1 = (7 + 2\sqrt{12})(7 - 2\sqrt{12}) \quad (9)$$

By using equation (9) and the value of z, we can write

$(4\alpha + \sqrt{12}\beta)(4\alpha - \sqrt{12}\beta) = (4a + \sqrt{12}b)^2(4a - \sqrt{12}b)^2(7 + 2\sqrt{12})(7 - 2\sqrt{12})$ Equating positive and negative terms we get,

$$\alpha = \alpha(a,b) = 28a^2 + 21b^2 + 48ab$$

$$\beta = \beta(a,b) = 32a^2 + 24b^2 + 56ab$$

Replacing the preceding variables of α and β in equation (2), the equivalent integer answers to (1) are provided by

$$x = x(a,b) = 60a^2 + 45b^2 + 104ab$$

$$y = y(a,b) = -4a^2 - 3b^2 - 8ab$$

$$z = z(a,b) = 16a^2 - 12b^2$$

OBSERVATION:

1) $y(a,1) - x(a,1) + 16T_{10,a} + 80Gno_a \equiv 0(\bmod 128)$
2) $z(a,1) - y(a,1) - x(a,1) + 40T_{4,a} + 48Gno_a \equiv 0(\bmod 102)$

3) $9z(1,1)+2y(1,1)$ is a nasty number
4) $10z(a,a)-4y(a,a)$ is a Perfect square
5) $3x(a,a)+4y(a,a)-567T_{4,a}=0$
6) $x(a,a)-y(a,a)-2z(a,a)$ is a cubical integer

C. Pattern: 3

Equation (3) can be written as

$$z^2+12\beta^2=16\alpha^2 \tag{10}$$

Assume, $\alpha=a^2+12b^2$ (11)

$$16=\frac{(4+i2\sqrt{12})(4-i2\sqrt{12})}{4} \tag{12}$$

In equations (11) and (12) in (10) we get,

$$z^2+12\beta^2=\frac{(4+i2\sqrt{12})(4-i2\sqrt{12})}{4}(a^2+7b^2)^2 \tag{13}$$

Equation (13) as,

$(z+i\sqrt{12}\beta)(z-i\sqrt{12}\beta)=(\frac{4+i2\sqrt{12}}{2})(\frac{4-i2\sqrt{12}}{2})(a+i\sqrt{12}b)^2(a-i\sqrt{12}b)^2$ Equating positive and negative terms we get,

$$z=z(a,b)=2a^2-24b^2-24ab$$

$$\beta=\beta(a,b)=a^2-12b^2+4ab$$

Replacing the preceding variables of z and β in equation (2), the equivalent answers to (1) are provided by

$$x=x(a,b)=2a^2+4ab$$

$$y=y(a,b)=24b^2-4ab$$

$$z=z(a,b)=2a^2-24b^2-24ab$$

OBSERVATION:

1) $x(a,1)-y(a,1)+z(a,1)-4\Pr_a+8Gno_a\equiv 0(\bmod 60)$
2) $2y(1,a)-z(1,a)-48\Pr_a-10Gno_a\equiv 0(\bmod 40)$
3) $x(2a,a)-y(2a,a)+88T_{4,a}=0$
4) $x(1,1)-2z(1,1)$ is a Perfect square
5) $x(a,a)-y(a,a)+14T_{4,a}=0$
6) $x(1,a)+z(a,1)-2T_{4,a}+10Gno_a\equiv 0(\bmod 32)$

D. Pattern: 4

Equation (3) as, $16\alpha^2-z^2=12\beta^2$ (14)

Write equation (14) as

$$(4\alpha + z)(4\alpha - z) = 12\beta \times \beta \quad (15)$$

Equation (15) is written in the form of a ratio as

$$\frac{4\alpha + z}{\beta} = \frac{12\beta}{4\alpha - z} = \frac{p}{q}$$

This is equivalent to the following two equations

$$4\alpha q + zq - \beta p = 0 \quad (16)$$

$$4\alpha p - zp - 12\beta q = 0 \quad (17)$$

By using cross-multiplication, we get

$$\alpha = \alpha(a,b) = p^2 + 12q^2$$

$$\beta = \beta(a,b) = 8pq$$

$$z = z(a,b) = 4p^2 - 48q^2$$

Replacing the preceding variables of α and β in equation (2), the equivalent integer answers to (1) are provided by

$$x = x(p,q) = p^2 + 12q^2 + 8pq$$

$$y = y(p,q) = p^2 + 12q^2 - 8pq$$

$$z = z(p,q) = 4p^2 - 48q^2$$

OBSERVATION:

1) $x(p,p) + y(p,p) + z(p,p)$ is a Perfect square

2) $2z(1,1) - 16y(1,1)$ is a cubical integer

3) $x(p,p) + y(p,p) + z(p,p) - 6\Pr_p \equiv 0 \pmod{30}$

4) $x(p,1) - y(p,1) + z(p,1) - 2T_{6,p} - 9Gno_p \equiv 0 \pmod{39}$

5) $x(p,p) + z(p,p) + 23p^2 = 0$

6) $x(2,p) - z(2,p) - 60T_{4,p} - 8Gno_p \equiv 0 \pmod{4}$

IV. CONCLUSION

In this paper, We have found an endless number of non-zero distinct integer solutions to the ternary quadratic Diophantine equation $2(x^2 + y^2) - 3xy = 16z^2$ To sum up, one can look for other solution patterns and their accompanying attributes.

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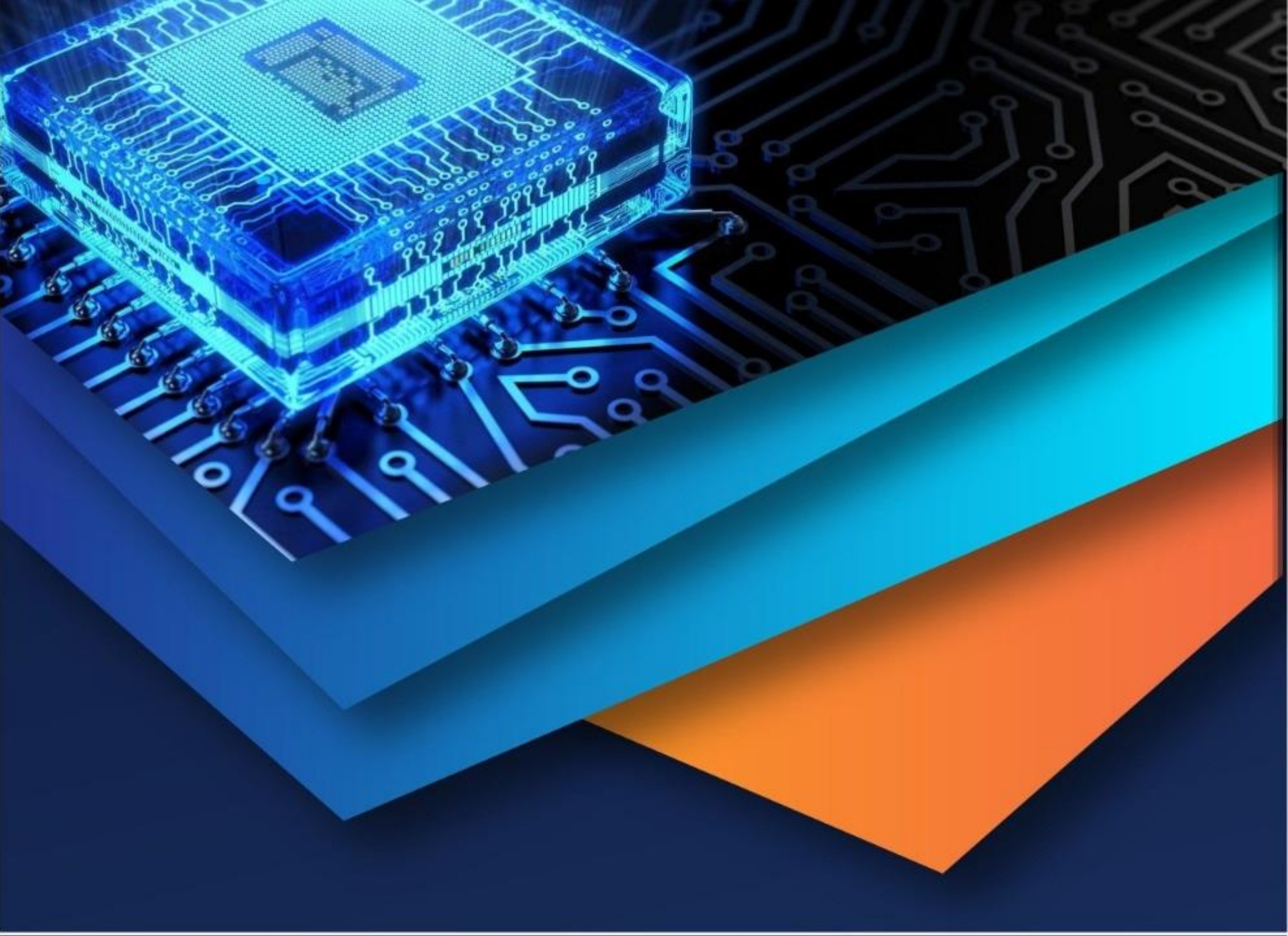

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