iJRASET

International Journal For Research in Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 9 Issue: X Month of publication: October 2021

DOI: https://doi.org/10.22214/ijraset.2021.38573

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.429
Volume 9 Issue X Oct 2021- Available at www.ijraset.com

Order Statistics of Additive Uniform Exponential Distribution

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Abstract: In this paper we investigated the order statistics by using Additive Uniform Exponential Distribution (AUED) proposed by Venkata Subbarao Uppu (2010).The probability density functions of rth order Statistics, lth moment of the rth order Statistic, minimum, maximum order statistics, mean of the maximum and minimum order statistics, the joint density function of two order statistics were calculated and discussed in detailed . Applications and several aspects were discussed
Keywords: Additive Uniform Exponential Distribution, Moments, Minimum order statistic, Maximum order statistic, Joint density of the order Statistics, complete length of service.

I. INTRODUCTION

In several data sets arising pollutant concentration analysis, demography, risk analysis, actuarial Statistics, manpower modeling, etc, the variable under study is a sum of two or more variables. For example, in manpower modeling the complete length of service of an employee in the organization varies according to two types of factors namely, semi committed and committed states (temporary and permanent) respectively. Generally an employee during semi committed state may stay with the organization in the same pattern throughout the period of that state, and the frequency distribution associated with that part is a random variable and follows a uniform distribution. Once the employee is made permanent and confirmed in the organization the rate of leaving is constant and the frequency distribution associated with this part is like an exponential distribution. Therefore the total duration of stay of an employee which is known as complete length of service of an employee in the organization is the sum of two random variables, which are distributed as uniform and exponential. Very little work has been reported in literature regarding the Additive Uniform Exponential Distribution, which has a tremendous potential in analyzing many data sets arising at places like, manpower planning and other domains. Using the Jacobean transformation of random variables the probability density function of the distribution is obtained. This distribution includes uniform and exponential distributions as particular cases for limiting values of the parameters.

A. Additive Uniform Exponential Distribution

The probability density function of additive uniform exponential distribution (AUED) is

$$f_X(x) = \frac{1}{a}\left[1 - e^{-\theta x}\right]; 0 \le X \le a$$
$$= \frac{e^{-\theta x}}{a}\left[e^{a\theta} - 1\right]; a \le X < \infty \qquad (1)$$

a and θ are the parameters of the distribution, $a > 0$ and $\theta > 0$

Its distribution function is

$$F_X(x) = \frac{x}{a} + \frac{\left(1 - e^{-\theta x}\right)}{a\theta}; \qquad 0 \le x \le a$$
$$= 1 + \frac{\left(e^{a\theta} - 1\right)e^{-\theta x}}{a\theta}; \qquad a \le t < \infty \qquad (2)$$

B. Order Statistics Of Additive Uniform Exponential Distribution

In this paper, we derive the distribution of extreme order statistics and the joint distribution of the order statistics and some properties. The distribution of extreme order statistics are very important for studying the inferences related to the maximum, minimum and median of the data sets. For example in manpower modeling the complete length of service of an employee is a random variable and studying its dynamics is very important for several operating policies regarding welfare and pensional benefits to find the probability distribution of the maximum duration of a state of an employee in an organization can be derived through order statistics.

Let U1:n ≤ U2:n≤ U3:n……….≤ Um:n be the order statistics obtained from a random sample of size n from AUED having the density function as given in (1)

The probability density function of the r[th] order statistics is given by (David 1981) is

$$f_{r:n}(u) = D_{r:n}[F(u)]^{r-1}[1-F(u)]^{n-r}f(u); \; -\infty < u < \infty \tag{3}$$

Where $D_{r:n} = \frac{n!}{(r-1)!(n-r)!}$ (4)

Substituting the values from equations (1) and (2)

$$f_{r:n}(u) = D_{r:n}\left[u\theta + e^{-\theta u} - 1\right]^{r-1}\left[(a-u)\theta - e^{-\theta u} + 1\right]^{n-r}\frac{\left(1-e^{-\theta u}\right)}{a^n\theta^{n-1}}$$

For $0 \le u \le a$ (5)

$$f_{r:n}(u) = \frac{D_{r:n}}{a^n\theta^{n-1}}\left[a\theta + e^{-\theta u}\left(1-e^{a\theta}\right)\right]^{r-1}\left[e^{-\theta u}\left(e^{a\theta}-1\right)\right]^{n-r+1} \text{ For } a \le u < \infty \tag{6}$$

Where $D_{r:n}$ is given in equation (3)

The l^{th} moment of the r[th] order statistic is

$$E(U^l{}_{r:n}) = \int_0^a u^l D_{r:n}\left[u\theta + e^{-\theta u} - 1\right]^{r-1}\left[(a-u)\theta + \left(1-e^{-\theta u}\right)\right]^{n-r}\frac{\left(1-e^{-\theta u}\right)}{a^n\theta^{n-1}}\,du$$

$$+ \frac{D_{r:n}}{a^n\theta^{n-1}}\left(e^{a\theta}-1\right)^{n-r+1}\int_a^\infty u^l\left[a\theta + e^{-\theta u}\left(1-e^{a\theta}\right)\right]^{r-1}e^{-\theta(n-r+1)}du$$

=

$$\sum_{s=0}^{n-r}\sum_{k=0}^{r-1}\sum_{m=0}^{n-r-s}\binom{n-1}{r}\binom{r-1}{k}\binom{n-r-s}{m}\frac{D_{r:n}}{a^n\theta^{n-1}}\,\theta^{n-k-1}(-1)^{k+m}a^{n-r-s-m}\int_0^a u^{m+l+r-k-1}\left(1-e^{-\theta u}\right)^{s+k+1}du$$

$$+ \frac{D_{r:n}}{a^n\theta^{n-1}}\left(e^{a\theta}-1\right)^{n-r+1}\int_a^\infty u^l\left[a\theta + e^{-\theta u}\left(1-e^{a\theta}\right)\right]^{r-1}e^{-\theta(n-r+1)}du$$

=

$$\sum_{s=0}^{n-r}\sum_{k=0}^{r-1}\sum_{m=0}^{n-r-s}\sum_{j=0}^{s+k+1}\sum_{i=0}^{\infty}\binom{n-1}{r}\binom{r-1}{k}\binom{s+k+1}{j}\binom{n-r-s}{m}\frac{D_{r:n}(j)^i}{a^n\theta^k(i)!}(-1)^{i+k+m+j}a^{n-r-s-m}$$

$$\int_0^a u^{i+m+l+r-k-1}du + \sum_{s=0}^{r-1}\binom{r-1}{s} + \frac{D_{r:n}}{a^n\theta^{n-1}}\left(e^{a\theta}-1\right)^{n-r+1}(a\theta)^{r-s-1}\int_a^\infty u^l\, e^{-\theta u(s+n-r+1)}du$$

And on simplification we get

$$\sum_{s=0}^{n-r}\sum_{k=0}^{r-1}\sum_{m=0}^{n-r-s}\sum_{j=0}^{s+k+1}\sum_{i=0}^{\infty}\binom{n-1}{r}\binom{r-1}{k}\binom{s+k+1}{j}\binom{n-r-s}{m}$$

$$\frac{D_{r:n}(j)^i}{a^n\theta^k(i)!}(-1)^{i+k+m+j}a^{2(i+n-k)+l-r-s-m-1} +$$

$$\sum_{s=0}^{r-1}\binom{r-1}{s} + \frac{D_{r:n}}{a^{n-r+s+1}\theta^{n-r+s}}\left(e^{a\theta}-1\right)^{n-r+s-1}\left[\frac{a^l e^{-a\theta(s+n-r+1)}}{\theta(s+n-r+1)} + \frac{l a^{l-1}e^{-a\theta(s+n-r+1)}}{\theta^2(s+n-r+1)^2}\right] \tag{7}$$

The probability density function of the first order statistic of AUED can be obtained by taking r=1 in equations (5) and (6)

$$f_{1:n}(u) = D_{1:n}\left[u\theta + e^{-\theta u} - 1\right]^{1-1}\left[(a-u)\theta - e^{-\theta u} + 1\right]^{n-1}\frac{\left(1-e^{-\theta u}\right)}{a^n\theta^{n-1}}$$

$$= n\left[(a-u)\theta - e^{-\theta u} + 1\right]^{n-1}\frac{\left(1-e^{-\theta u}\right)}{a^n\theta^{n-1}} \quad \text{For } 0 \le u \le a \tag{8}$$

And

$$f_{1:n}(u) = \frac{D_{1:n}}{a^n\theta^{n-1}}\left[a\theta + e^{-\theta u}\left(1-e^{a\theta}\right)\right]^{1-1}\left[e^{-\theta u}\left(e^{a\theta}-1\right)\right]^{n-1+1}$$

$$= \frac{n}{a^n\theta^{n-1}}\left[e^{-\theta u}\left(e^{a\theta}-1\right)\right]^n \quad \text{For } a \le u < \infty \tag{9}$$

The mean of the first order statistic is given by

$$E(U_{(1)}) = \int_0^a nu[(a-u)\theta - e^{-\theta u} + 1]^{n-1} \frac{(1-e^{-\theta u})}{a^n\theta^{n-1}}\, du + \int_a^\infty \frac{nu}{a^n\theta^{n-1}}[e^{-\theta u}(e^{a\theta}-1)]^n\, du$$

$$=\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{n\theta^{n-r-1}}{a^n\theta^{n-1}}\int_0^a u[1-e^{-\theta u}]^{r+1}(a-u)^{n-r-1}du + \frac{n(e^{a\theta}-1)^n}{a^n\theta^{n-1}}\int_a^\infty ue^{-n\theta u}\, du$$

On simplification we get

$$E(U_{(1)}) = \sum_{r=0}^{n-1}\sum_{s=0}^{r+1}\sum_{k=0}^{n-r-1}\binom{n-1}{r}\binom{r+1}{s}\binom{n-r-1}{k}\frac{n(-1)^{k+1}}{a^{r+k+1}\theta^r}\left[\frac{a^{k+2}}{k+2} - \frac{\theta s a^{k+3}}{k+3}\right]$$

$$+\frac{(e^{a\theta}-1)^n}{a^n\theta^n}\left[ae^{-an\theta} + \frac{e^{-an\theta}}{n\theta}\right] \qquad (10)$$

The probability density function of the highest order statistic of AUED can be obtained by taking r = n in equations (5) and (6)

$$f_{n:n}(u) = D_{n:n}[u\theta + e^{-\theta u} - 1]^{n-1}[(a-u)\theta - e^{-\theta u} + 1]^{n-n}\frac{(1-e^{-\theta u})}{a^n\theta^{n-1}}$$

$$= n[u\theta + e^{-\theta u} - 1]^{n-1}\frac{(1-e^{-\theta u})}{a^n\theta^{n-1}} \text{ ; For } 0 \le u \le a \qquad (11)$$

And

$$f_{n:n}(u) = \frac{D_{n:n}}{a^n\theta^{n-1}}[a\theta + e^{-\theta u}(1-e^{a\theta})]^{n-1}[e^{-\theta u}(e^{a\theta}-1)]^{n-n+1}$$

$$= \frac{n}{a^n\theta^{n-1}}[a\theta + e^{-\theta u}(1-e^{a\theta})]^{n-1}e^{-\theta u}(e^{a\theta}-1); \text{ For } a \le u < \infty \qquad (12)$$

Where $D_{n:n}$ is calculated by using equation (4)

The mean of the highest order statistic is given by

$$E(U_{(n)}) = \int_0^a nu[u\theta + e^{-\theta u} - 1]^{n-1}\frac{(1-e^{-\theta u})}{a^n\theta^{n-1}}\, du + \int_a^\infty \frac{nu}{a^n\theta^{n-1}}[a\theta + e^{-\theta u}(1-e^{a\theta})]^{n-1}e^{-\theta u}(e^{a\theta}-1)\, du$$

And on simplification we get

$$\sum_{r=0}^{n-1}\sum_{s=0}^{r+1}\sum_{k=0}^{\infty}\binom{n-1}{r}\binom{r+1}{s}\frac{n(-1)^{r+s+k}s^k}{a^n\theta^{r-k}}\left[\frac{a^{n-r+k+1}}{n-r+k+1}\right] + \sum_{r=0}^{n-1}\frac{n(-1)^r(e^{a\theta}-1)^{r+1}}{a^{r+1}\theta^r}\left[\frac{ae^{-ar\theta}}{-r\theta} + \frac{e^{-ar\theta}}{\theta^2 r^2}\right] \qquad (13)$$

The joint density function of the order statistics $U_{m:n}$ and $U_{s:n}$(m< S) is given by

$$f(u,v)_{m,s:n} = D_{m,s:n}[F(U)]^{m-1}[F(V)-F(U)]^{s-m-1}[1-F(V)]^{n-s}f(u)f(v) \qquad (14)$$

Where $D_{r:n} = \frac{n!}{(r-1)!(n-r)!}$

Here ,We have three cases based up on the sample space of the variates they are

(i) $0 \le U_m < U_s \le a$

(ii) $0 \le U_m \le a; a \le U_s < \infty$

(iii) $a \le U_m < U_s < \infty$

For case (i) the joint probability density function of the order statistics $U_{m:n}$ and $U_{s:n}$ is

$$f(u,v)_{m,s:n} = D_{m,s:n}[u\theta + e^{-\theta u} - 1]^{m-1}[v\theta + e^{-\theta v} - u\theta - e^{-\theta u}]\frac{(1-e^{-\theta u})(1-e^{-\theta v})}{a^n\theta^{n-2}} \qquad (15)$$

Where $D_{m,s:n}$ is given in equation (4)

For case (ii)

We have

$$f(u,v)_{m,s:n} = \frac{D_{m,s:n}}{a^n\theta^{n-2}}[u\theta + e^{-\theta u} - 1]^{m-1}[(a-u)\theta + e^{-\theta v}(e^{a\theta}-1) + (1-e^{-\theta u})]^{s-m-1}(e^{a\theta}-1)^{n-s+1}(-1)^{n-s}e^{-\theta v(n-s+1)}$$

(16)

Where $D_{m,s:n}$ is given in equation (4)

For case (iii)

$$f(u,v)_{m,s:n} = \frac{D_{m,s:n}}{a^n\theta^{n-2}}\left[a\theta + e^{-\theta u}\left(1-e^{a\theta}\right)\right]^{m-1}\left[\left(e^{-\theta u}-e^{-\theta v}\right)\right]^{s-m-1} \left(e^{a\theta}-1\right)^{s-m}\left[a\theta - e^{-\theta v}\left(1-e^{a\theta}\right)\right]^{n-s}e^{-\theta v}\left(1-e^{-\theta u}\right) \quad (17)$$

Where $D_{m,s:n}$ is given in equation (4)

The joint density function of the smallest and highest order statistics can be obtained by taking m=1 and S = n in the equations (15),(16),(17)

Then for case (i)

$$f(u)_{1,n:n} = D_{1,n:n}\left[(v-u)\theta + e^{-\theta v} - e^{-\theta u}\right]^{n-2}\frac{(1-e^{-\theta u})(1-e^{-\theta v})}{a^n\theta^{n-2}} \quad (18)$$

For case (ii)

$$f(u)_{1,n:n} = D_{1,n:n}\left[(a-u)\theta + e^{-\theta v}\left(e^{a\theta}-1\right) + \left(1-e^{-\theta u}\right)\right]^{n-2}\frac{e^{-\theta v}(e^{a\theta}-1)}{a^n\theta^{n-2}} \quad (19)$$

For case (iii)

$$f(u)_{1,n:n} = D_{1,n:n}\left[e^{a\theta}-1\right]^{n-1}\left[e^{-\theta u}-e^{-\theta v}\right]^{n-2}\frac{e^{-\theta v}(1-e^{-\theta u})}{a^n\theta^{n-2}} \quad (20)$$

II. CONCLUSIONS

The above order statistics are very useful in manpower planning models, especially the minimum and maximum order statistics are used to calculate the pensionable benefits of an employee in an organization by treating the complete length of service as a random variable which is additive in nature. Order statistics were employed in many ways in acceptance sampling. First-order statistics are used to improve the robustness of sampling plans by variables. In life testing, these are much useful to shorten testing times to produce many lifetime distributions. In actuarial sciences, these have tremendous potential in joint life insurance aspects to calculate the distribution of life span and insurance risk. Order statistics are concerned with the ranks as well as the magnitude of the observations we can use them in the grouping of continuous data into frequency classification.

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