iJRASET
International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 10 **Issue:** VIII **Month of publication:** August 2022

DOI: https://doi.org/10.22214/ijraset.2022.46147

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.538
Volume 10 Issue VIII Aug 2022- Available at www.ijraset.com

Properties of the Ternary Cubic Equation $5x^2 - 3y^2 = z^3$ ∑

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Abstract: To identify its different integral non-zero solutions, the ternary cubic equation $5x^2 - 3y^2 = z^3$ is taken into consideration. Different integral solution patterns to the ternary cubic equation under consideration are obtained in each pattern by using the linear transformation and the method of factorization; interesting relationships between the solutions and some polygonal numbers, such as pyramidal and central pyramidal numbers, are also displayed.
Keywords: Diophantine equations, Ternary equation, Cubic Equation with Three Unknowns, Integral Solutions

I. INTRODUCTION

Number theory, which is used to explain anything that can be quantified, is the language of patterns and relationships. A polynomial equation called a Diophantine equation can only have integers as solutions. Theories of numbers were featured in [1–3]. A unique Pythagorean triangle problem and its integral solutions are featured in [4, 5]. Higher order equations are taken into account for integral solutions in [6–10].

The non-homogeneous cubic equation with three unknowns represented by the equation is discussed in this communication, and in particular, a few intriguing relationships between the solutions are highlighted.

A. Notations

$T_{m,a}$: Polygonal Number of rank a with side m
Gno_a: Gnomonic Number of rank a
$Star_a$: Star Number of rank a
O_a: Octahedral Number of rank a
P_a^m: Pyramidal Number of rank a with sides m
SO_a: Stella Octangula Number of rank a
CC_a: Centered Cube Number of rank a
CS_a: Centered Square Number of rank a
RD_a: Rhombic Dodecagonal Number of rank a
TO_a: Truncated Octahedral Number of rank a

II. METHOD OF ANALYSIS

A non-zero integral solution to the Cubic equation can be found by

$$5x^2 - 3y^2 = z^3 \qquad (1)$$

Upon switching the transformations,

$$x = X + 3T,\ y = X + 5T,\ z = 2Z \qquad (2)$$

in (1) leads to, $X^2 - 15T^2 = 4Z^3$ (3)

Below, we present illustration of distinct integer non-zero patterns (1)

A. Pattern: 1

Assume $Z = z(a,b) = a^2 - 15b^2$ (4)

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Where a and b are positive integers.

And write $4=(8+2\sqrt{15})(8-2\sqrt{15})$ (5)

Using the factorization approach, replacing (3) with (4) and (5), $(X+\sqrt{15}T)(X-\sqrt{15}T)=(8+2\sqrt{15})(8-2\sqrt{15})(a+\sqrt{15}b)^3(a-\sqrt{15}b)^3$

Comparing real and imaginary elements while equating similar phrases, $X=8a^3+360ab^2+90a^2b+450b^3$

$T=2a^3+90ab^2+24a^2b+120b^3$

The appropriate integer solutions of equation (1) are provided by substituting the above mentioned values of X and T into equation

$x=x(a,b)=14a^3+630ab^2+162a^2b+810b^3$

(2) $y=y(a,b)=18a^3+810ab^2+210a^2b+1050b^3$

$z=z(a,b)=2a^2-30b^2$

Properties:

1. $-7z(a,a)$ and $-28z(a,a)$ are Perfect Squares
2. $-x(1,1)+y(1,1)-z(1,1)$ is a Harshad Number
3. $y(a,1)-x(a,1)-6P_a^6-star_a-2T_{29,a}-T_{26,a}\equiv 239(\bmod 123)$
4. $y(a,1)-x(a,1)-124z(a,1)-2SO_a-91Gno_a\equiv 0(\bmod 4051)$
5. $-y(a,1)-z(a,1)+18P_a^3+4Star_a\equiv 1076(\bmod 852)$
6. $x(a,1)+y(a,1)-16z(a,1)-2TO_a-29T_{30,a}\equiv 2352(\bmod 1769)$
7. $-x(a,1)+y(a,1)-23z(a,1)+12P_a^5+T_{18,a}\equiv 930(\bmod 187)$

B. Pattern: 2

The modified form of equation (3) is

$X^2-15T^2=4Z^3*1$ (6)

Put 4 in as, $4=(8+2\sqrt{15})(8-2\sqrt{15})$ (7)

and $1=(4+\sqrt{15})(4-\sqrt{15})$ (8)

When (7) and (8) are substituted in equation (6) and the process of factorization is used, as described in Pattern 1, the corresponding

$$x=x(a,b)=110a^3+4950ab^2+1278a^2b+6390b^3$$

integer solutions of (1) are represented by $y=y(a,b)=142a^3+6390ab^2+1650a^2b+8250b^3$

$$z=z(a,b)=2a^2-30b^2$$

Properties:

1. $y(1,1)-x(1,1)+z(1,1)-22T_{3,4}$ is a nasty number
2. $x(1,1)+y(1,1)-26T_{3,4}$ is a perfect square number
3. $y(1,1)-x(1,1)+11z(1,1)-T_{8,3}$ Is a cubic number
4. $y(a,1)-x(a,1)-8RD_a-72T_{15,a}-886Gno_a\equiv 0(\bmod 2762)$
5. $y(a,1)-x(a,1)-186z(a,1)-48O_a-712Gno_a\equiv 0(\bmod 8152)$

6. $y(a,1) - x(a,1) - 2TO_a - 47T_{22,a} \equiv 1392(\mathrm{mod}\,1815)$
7. $y(a,1) - 825x(a,1) - 213O_a \equiv 33000(\mathrm{mod}\,6319)$
8. $2x(a,1) - 426z(a,1) - 110SO_a - 284T_{26,a} - 6567Gno_a \equiv 0(\mathrm{mod}\,6567)$

C. Pattern: 3

Write $1 = (31+8\sqrt{15})(31-8\sqrt{15})$ (9)

By replacing (7) and (9) in equation (6) and factorising the result using the steps in Pattern 1, the corresponding integer solutions of (1) are represented by

$$x = x(a,b) = 866a^3 + 38970ab^2 + 10062a^2b + 50310b^3$$
$$y = y(a,b) = 1118a^3 + 50310ab^2 + 12990a^2b + 64950b^3$$
$$z = z(a,b) = 2a^2 - 30b^2$$

Properties:

1. $y(a,1) - 16z(a,1) - 2236P_a^5 - 1184T_{22,a} - 30483Gno_a \equiv 0(\mathrm{mod}\,95913)$
2. $y(a,1) - x(a,1) + z(a,1) - 189HO_a - 827T_{10,a} - 5796Gno_a \equiv 0(\mathrm{mod}\,23056)$
3. $\frac{1}{10}(x(1,1) - y(1,1))$ is a square number
4. $x(1,b) + y(1,b) + 23z(1,b) - 138312P_b^7 - 3239Star_b - 44295Gno_b \equiv 0(\mathrm{mod}\,42086)$
5. $y(1,1) - x(1,1) + 24T_{3,4}$ is a nasty number
6. $x(a,1) - 16z(a,1) - 4336CC_a - 11329CS_a + 22658T_{3,a} \equiv 39894(\mathrm{mod}\,71658)$
7. $-x(a,1) + y(a,1) - 23z(a,1) - 63RD_a - 239T_{30,a} \equiv 14013(\mathrm{mod}\,14195)$
8. $y(a,1) - x(a,1) + 19z(a,1) - 126SO_a - 1483T_{6,a} \equiv 14070(\mathrm{mod}\,12949)$

Note:

Additionally, 4 and 1 might be written as

$$4 = (62+16\sqrt{15})(62-16\sqrt{15}$$

$$1 = \frac{(8+\sqrt{15})(8-\sqrt{15})}{49}$$

In relation to these options, one may find many patterns of solutions of (1)

III. CONCLUSION

Three unique patterns of non-zero distinct integer solutions to the given non-homogeneous problem $5x^2 - 3y^2 = z^3$ are shown in this paper.

For various options of cubic Diophantine equations, additional patterns of non-zero integer unique solutions and their corresponding characteristics may be found.

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