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Solution of Exponential Diophantine Equation Involving Fermat Primes

C. Saranya[1], A. Swetha Shree[2]
[1]Associate Professor, [2]PG Student, PG and Research Department of Mathematics, Cauvery College for Women (Autonomous) (Affiliated to Bharathidasan University), Tiruchirapalli, Tamilnadu, India

Abstract: This paper investigates the solutions of specific exponential Diophantine equations involving Fermat primes. Through a structured case-by-case analysis, integer solutions are identified for equations of the form $a^x + b^y = z^2$, where x, y, z are non-negative integers. Theorems and proofs are provided to demonstrate the validity of these solutions. The study confirms that certain triplets, such as (1,0,2), (2,2,5), and others, satisfy the given equations under distinct mathematical constraints. These results extend the theoretical framework of Diophantine analysis and may have implications for cryptographic applications and computational number theory.
Keywords: Exponential Diophantine Equation, Fermat Prime, Integer Solutions, Catalan Conjecture.

I. INTRODUCTION

Diophantine equations, named after the ancient Greek mathematician Diophantus, are equations where only integer solutions are sought $[1]$. Among these, exponential Diophantine equations have a particular structure where exponents are variables, making them complex and intriguing in number theory $[5]$. This paper focuses on a specific class of exponential Diophantine equations that involve Fermat primes number. Fermat prime numbers of the form $2^{2^n}+1$ $(e.g.,3,5,17,257)$. In $[8]$, By analyzing these equations under different cases, we determine integer solutions that satisfy their conditions. The findings contribute to the broader understanding of number theory, particularly in identifying patterns and constraints in such equations.

II. DEFINITION

A. Catalan's Conjecture

Catalan's conjecture (or Mihăilescu's theorem) is a theorem in number theory that was conjectured by the mathematician Eugène Charles Catalan in 1844 and proven in 2002 by Preda Mihăilescu at Paderborn University. The integers 2^3 and 3^2 are two perfect powers(that is, powers of exponent higher than one) of natural numbers whose values (8 and 9, respectively) are consecutive. The theorem states that this is the *only* case of two consecutive perfect powers. That is to say, that The Diophantine equation $a^x+b^y=1$ has unique integer solution with min $\{a, b, x, y\}>1$. The solution (a, b, x, y) is (3, 2, 2, 3).

1) Theorem: 1

(1,0,2) and (2,2,5) are the solution of the exponential Diophantine equation

$$3^x + 4^y = z^2 \quad (1)$$

where $x,\ y$ & z are non-negative integers.

Proof:

We will divide the proof under 3 cases.

Case 1

Suppose $x = 0$, then $81^x + 12^y = z^2$ becomes

$$1+4^Y = Z^2$$

Let $z-1 = 2^u$, (2)

where u are non-negative integers.

Then $z+1 = 2^{2y-u}$ (3)

Using (2) & (3), we get, $2^{2y-u} - 2^u = 2$

$2^u[2^{2y-2u}-1]=2$

$\Rightarrow u=1$

Then $2^{2y-2}=3$. Which is impossible for positive values of y

so that $x \neq 0$.

Case 2

Suppose $y=0$ & write (1) as $3^x+1=z^2$

$z^2-1=3^x$

Let $z-1=3^v$, (4)

Where v is a non-negative integer.

Then $z+1=3^{x-v}$ (5)

Using (4) & (5), we get, $3^{x-v}-3^v=2$

$3^v[3^{x-2v}-1]=2$

$\Rightarrow v=0$

If y = 0 then x = 1, so that z = 2

Hence (0,1,2) is the solution of $81^x+12^y=z^2$

Case 3

Suppose $x=2$, rewrite (1) as $3^x+2^{2y}=z^2$

$z^2-2^{2y}=3^w$

Let $z-2^y=3^w$, (6)

where w are non-negative integers.

then $z+2^y=3^{x-w}$ (7)

Using (6) & (7),

$3^{x-w}-3^w=2^{y+1}$

$\Rightarrow w=0$

then $3^2-1=2^{y+1}$

If $x=2$ then $y=1$, so that $z=5$

Hence $(2\ 1,5)$ is also the solution of $3^x+4^y=z^2$.

In general, (0,1,2) and (2,1,5) are the solution of the exponential Diophantine equation $3^x+4^y=z^2$

2) Theorem:2

(1,1,3) is the solution of the exponential Diophantine equation

$$5^x+4^y=z^2 \quad (8)$$

where $x,\ y$ & z are non-negative integers.

Proof:

We will divide the proof under 3 cases.

Case 1

Suppose $x=0$, then $5^x+4^y=z^2$ becomes

$1+4^Y=Z^2$

Let $z-1=2^{u}$, (9)
where u are non-negative integers.

Then $z+1=2^{2y-u}$ (10)

Using (9) & (10), we get, $2^{2y-u}-2^{u}=2$

$$2^{u}[2^{2y-2u}-1]=2$$

$$\Rightarrow u=1$$

Then $2^{2y-2}=3$. Which is impossible for positive values of y

so that $x\neq 0$.

Case 2

Suppose $y=0$ & write (8) as $5^{x}+1=z^{2}$

$$z^{2}-1=5^{x}$$

Let $z-1=5^{v}$, (11)
where v is a non-negative integer.

Then $z+1=5^{x-v}$ (12)

Using (11) & (12), we get, $5^{x-v}-5^{v}=2$

$$5^{v}[5^{x-2v}-1]=2$$

$$\Rightarrow v=0$$

Then $5^{x}=3$. Which is impossible for positive values of x

so that $y\neq 0$.

Case 3

Suppose $x\geq 2$, rewrite (8) as $5^{x}+2^{2y}=z^{2}$

$$z^{2}-2^{2y}=5^{w}$$

Let $z-2^{y}=5^{w}$, (13)
where w are non-negative integers.

then $z+2^{y}=5^{x-w}$ (14)

Using (13) & (14),

$$5^{x-w}-5^{w}=2^{y+1}$$

$$\Rightarrow w=0$$

then $5^{x}-1=2^{y+1}$

If $x=1$ then $y=1$, so that $z=3$

Hence $(1,1,3)$ is also the solution of $5^{x}+4^{y}=z^{2}$.

3) Theorem:3

(1,3,9) is the solution of the exponential Diophantine equation

$$17^{x}+4^{y}=z^{2} \quad (15)$$

where $x,\ y$ & z are non-negative integers.

Proof:
We will divide the proof under 3 cases.

Case 1

Suppose $x=0$, then $17^x+4^y=z^2$ becomes

$$1+4^Y=Z^2$$

Let $z-1=2^u$, (16)

where u are non-negative integers.

Then $z+1=2^{2y-u}$ (17)

Using (16) & (17), we get, $2^{2y-u}-2^u=2$

$$2^u[2^{2y-2u}-1]=2$$

$$\Rightarrow u=1$$

Then $2^{2y-2}=3$. Which is impossible for positive values of y

so that $x\neq 0$.

Case 2

Suppose $y=0$ & write (15) as $17^x+1=z^2$

$$z^2-1=17^x$$

Let $z-1=17^v$, (18)

where v is a non-negative integer.

Then $z+1=17^{x-v}$ (19)

Using (18) & (19), we get, $17^{x-v}-17^v=2$

$$17^v[17^{x-2v}-1]=2$$

$$\Rightarrow v=0$$

Then $17^x=3$. Which is impossible for positive values of x

so that $y\neq 0$.

Case 3

Suppose $x\geq 2$, rewrite (15) as $17^x+2^{2y}=z^2$

$$z^2-2^{2y}=17^w$$

Let $z-2^y=17^w$, (20)

where w are non-negative integers.

then $z+2^y=17^{x-w}$ (21)

Using (20) & (21),

$$17^{x-w}-17^w=2^{y+1}$$

$$\Rightarrow w=0$$

then $17^x-1=2^{y+1}$

If $x=1$ then $y=3$, so that $z=7$

Hence $(1,3,9)$ is also the solution of $17^x+4^y=z^2$.

4) Theorem:4

(1,7,129) is the solution of the exponential Diophantine equation

$$257^x+4^y=z^2 \quad (22)$$

where $x,\ y\ \&\ z$ are non-negative integers.

Proof:

We will divide the proof under 3 cases.

Case 1:

Suppose $x=0$, then $257^x+4^y=z^2$ becomes

$$1+4^Y=Z^2$$

Let $z-1=2^u$, (23)

where u are non-negative integers.

Then $z+1=2^{2y-u}$ (24)

Using (23) & (24), we get, $2^{2y-u}-2^u=2$

$$2^u[2^{2y-2u}-1]=2$$

$$\Rightarrow u=1$$

Then $2^{2y-2}=3$. Which is impossible for positive values of y

so that $x\neq 0$.

Case 2:

Suppose $y=0$ & write (22) as $257^x+1=z^2$

$$z^2-1=257^x$$

Let $z-1=257^v$, (25)

where v is a non-negative integer.

Then $z+1=2^{x-v}$ (26)

Using (26) & (25), we get, $257^{x-v}-257^v=2$

$$257^v[257^{x-2v}-1]=2$$

$$\Rightarrow v=0$$

Then $257^x=3$. Which is impossible for positive values of x

so that $y\neq 0$.

Case 3:

Suppose $x\geq 2$, rewrite (22) as $257^x+2^{2y}=z^2$

$$z^2-257^{2y}=5^w$$

Let $z-2^y=257^w$, (28)

where w are non-negative integers.

then $z+2^{y}=257^{x-w}$ (29)

Using (28) & (29),

$$257^{x-w}-257^{w}=2^{y+1}$$

$$\Rightarrow w=0$$

then $257^{x}-1=2^{y+1}$

If $x=1$ then $y=7$, so that $z=129$

Hence $(1,7,129)$ is also the solution of $257^{x}+4^{y}=z^{2}$.

III. CONCLUSION

The study successfully determines integer solutions for a class of exponential Diophantine equations involving Fermat primes. By dividing the proofs into systematic cases, we demonstrate the constraints under which such equations hold true. The solutions obtained contribute to the ongoing research in Diophantine analysis and provide a foundation for further exploration in number theory. Future research may focus on generalizing these findings to broader classes of equations or exploring their applications in computational fields.

REFERENCES

[1] G.Jeyakrishnan and G.Komahan, “On the diophantine equation $128^{x}+196^{y}=z^{2}$”, Acta Ciencia indica mathematics, vol 2, 195-196, 2016.

[2] G.Jeyakrishnan and G.Komahan, “On the diophantine equation $128^{x}+961^{y}=z^{2}$”, International Journal for Innovative Research in science & Technology, vol 3, issue 9, 119-120, Feb 2017.

[3] Ivan Niven, Herbert, S.Zuckerman and Hugh L.Montgomery, “An Introduction to the theoryof Numbers”, John Wiley and Sons Inc, New York 2004.

[4] L.K.Hua, “Introduction to the Theory of Numbers”, Springer-Verlag, Berlin-New york, 1982.

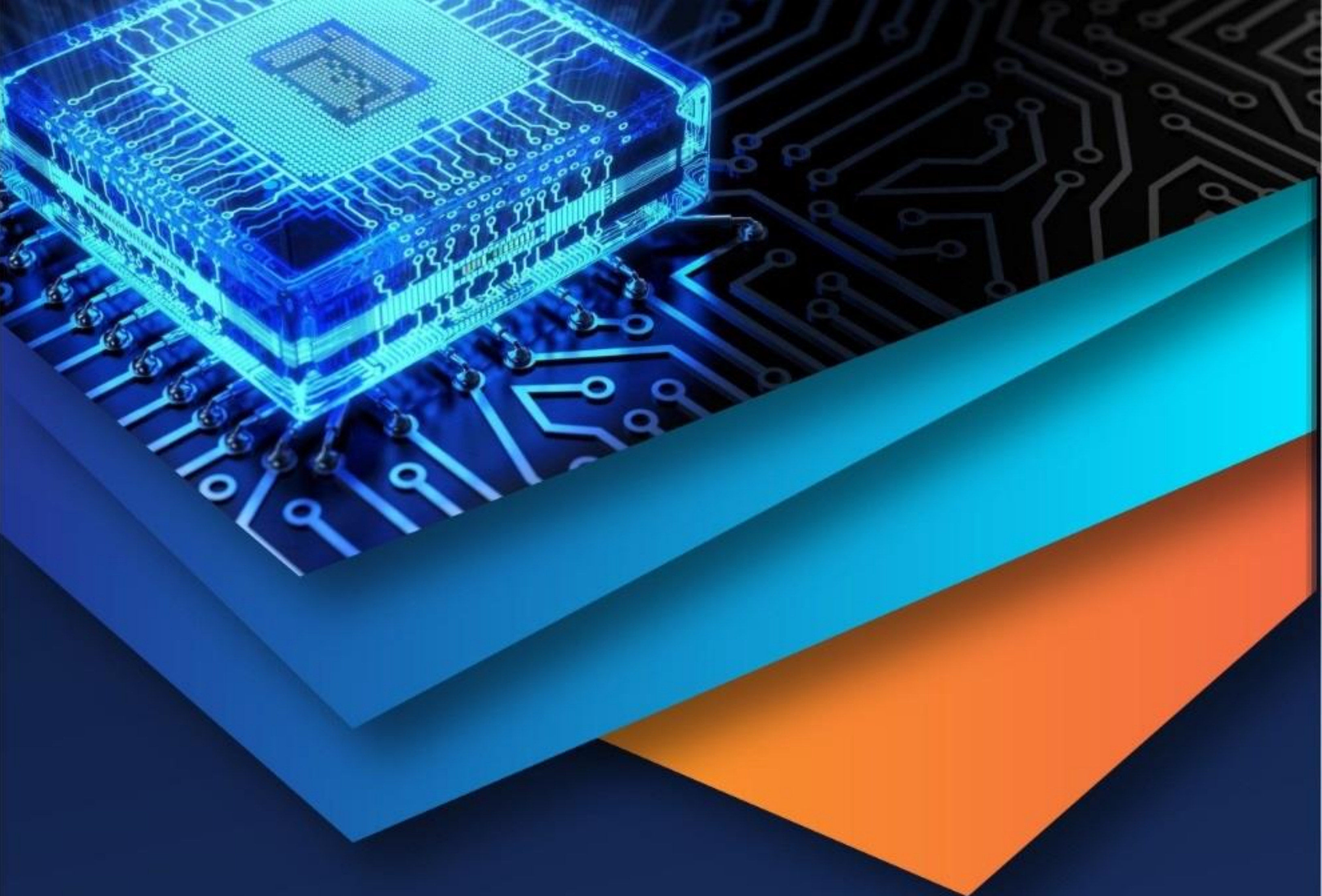
[5] P.Saranya and G.Janaki, “On the exponential diophantine equation $36^{x}+3^{y}=z^{2}$”, International Research Journal of Engineering and Technology, vol 4, issue 11, 1042-1044, Nov 2017

[6] R.D.Carmichael, “History of theory of numbers and diophantine analysis”, Dover Publication, New york, 1959.

[7] Somchitchotchaisthit, “On the diophantine equation $4^{x}+p^{y}=z^{2}$, where p is a prime number”, American Jr. of mathematics and sciences, vol-1, issue 1, Jan 2012.

[8] Saranya, C., and G. Yashvandhini. "Integral Solutions of an Exponential Diophantine Equation ." International Journal of Scientific Research in Mathematical and Statistical Sciences, vol. 9, no. 4, 2022, pp. 50-52

[9] Saranya, C., Janani M., and Salini R. "Integer Solutions of Some Exponential Diophantine Equations." The Mathematics Education, vol. 58, no. 1, Mar. 2024.



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