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Solutions of Pell's Equation Involving Wilson's Primes

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Abstract: The main goal of this study is to find non-trivial integer solutions to Pell's equation involving Wilson's primes. We obtain a series of unique integer solution and construct recurrence linkage between them using Brahmagupta's lemma. The derived solutions and their relationships are shown numerically, illustrating the function of Wilson's primes in producing these solutions.
Keywords: Brahmagupta lemma, Wilson's primes, recurrence relations, Diophantine equation, Pell's equation.

I. INTRODUCTION

A central theme in number theory is the study of Diophantine equations, which seek integer solutions to polynomial equations. Among them, Pell's equation $x^2 = Dy^2 + 1$. where D is a positive square-free integer, occupies a fundamental position due to its deep connections with quadratic fields, continued fractions, and algebraic number theory [1,3-7,9&11]. The equation admits infinitely many integer solutions whenever a fundamental non-trivial solution exists [2,8,10 &12]. Over the centuries, several powerful techniques have been developed to determine these solutions, including continued fraction expansions and algebraic methods. One of the most elegant classical tools in generating solutions of Pell's equation is Brahmagupta's lemma, which provides a composition law for solutions. This lemma enables the recursive construction of infinitely many solutions from a known pair, revealing the multiplicative structure underlying the solution set. The lemma not only demonstrates the richness of the equation's algebraic framework but also establishes recurrence relations that govern the growth of its solutions.

In recent developments, special classes of prime numbers have been employed to construct and characterize solutions of Diophantine equations. In particular, Wilson's primes, defined by the congruence $(p-1)! \equiv -1 (\bmod p^2)$ form a rare and remarkable subclass of primes arising from Wilson's theorem. Their exceptional modular properties and strong arithmetic constraints make them suitable candidates for generating structured integer solutions. The interplay between the factorial congruence condition of Wilson's primes and the quadratic form of Pell's equation provides a novel framework for deriving non-trivial solution sequences.

Motivated by these connections, the present study investigates Pell's equation through the combined application of Wilson's primes and Brahmagupta's lemma. By exploiting the arithmetic properties of Wilson's primes, we construct recurrence relations and establish the existence of distinct non-zero integer solutions. The theoretical results are substantiated with illustrative numerical examples, thereby contributing to the broader understanding of structured solution generation in classical Diophantine equations.

II. METHOD OF ANALYSIS

1) Theorem

The sequence of non-zero distinct integer solution to the Pell's equation

$$x^2 - Dy^2 = z^2 \qquad (1)$$

where, $D = W^2 - 1$, z is an integer and W is a Wilson's primes given by

$$x_s = \frac{z}{2}\left(\sqrt{D+1}\,f_s + \sqrt{D}\,g_s\right)$$

$$y_s = \frac{z}{2\sqrt{D}}\left(\sqrt{D}\,f_s + \sqrt{D+1}\,g_s\right) \qquad s=0,1,2\ldots$$

and their recurrence relations are

$$x_{s+2} - 2\sqrt{D+1}x_{s+1} + x_s = 0$$

$$y_{s+2} - 2\sqrt{D+1}y_{s+1} + y_s = 0 \qquad s=0,1,2...$$

Proof:

The initial solution of (1) is (x_0, y_0) is given by $x_0 = z\sqrt{D+1}, y_0 = z.$

To find the other solutions of (1), consider the Pell's equation

$$x^2 = D y^2 + 1$$

whose initial solution $(\tilde{x}_s, \tilde{y}_s)$ is given by

$$\tilde{x}_s = \frac{1}{2} f_s$$

$$\tilde{y}_s = \frac{1}{2\sqrt{D}} g_s \qquad s=0,1,2...$$

where $f_s = \left(\sqrt{D+1} + \sqrt{D}\right)^{s+1} + \left(\sqrt{D+1} - \sqrt{D}\right)^{s+1}$

$$g_s = \left(\sqrt{D+1} + \sqrt{D}\right)^{s+1} - \left(\sqrt{D+1} - \sqrt{D}\right)^{s+1},$$

Applying Brahmagupta's lemma between the solutions

$$(x_0, y_0) \text{ and } (\tilde{x}_s, \tilde{y}_s),$$

the sequence of non-zero distinct integer solutions to (1) are

$$x_s = \frac{z}{2}\left(\sqrt{D+1} f_s + \sqrt{D} g_s\right)$$

$$y_s = \frac{z}{2\sqrt{D}}\left(\sqrt{D} f_s + 2\sqrt{D+1} g_s\right) \qquad s=0,1,2...$$

and their recurrence relations are found to be

$$x_{s+2} - 2\sqrt{D+1}x_{s+1} + x_s = 0$$

$$y_{s+2} - 2\sqrt{D+1}\, y_{s+1} + y_s = 0, \qquad s=0,1,2\ldots$$

2) Illustration

The solutions of Pell's equation involving Wilson's Primes and its corresponding recurrence relations are presented in the table below:

Wilson's primes W	Integer z	Pell's equation	Sequence of Integer Solutions	Recurrence relation
5	2	$x^2 = 24y^2 + 4$	$x_s = 5f_s + \sqrt{24}g_s$ $y_s = \frac{1}{\sqrt{24}}\left(\sqrt{24}\, f_s + 5g_s\right)$	$x_{s+2} - 10x_{s+1} + x_s = 0$ $y_{s+2} - 10y_{s+1} + y_s = 0$
13	4	$x^2 = 168y^2 + 16$	$x_s = 2\left(13f_s + \sqrt{168}g_s\right)$ $y_s = \frac{2}{\sqrt{168}}\left(\sqrt{168}f_s + 13g_s\right)$	$x_{s+2} - 26x_{s+1} + x_s = 0$ $y_{s+2} - 26y_{s+1} + y_s = 0$

563	6	$x^2 = 316968y^2 + 36$	$x_s = 3(563 f_s + \sqrt{316968} g_s)$ $y_s = \frac{3}{\sqrt{316968}}(\sqrt{316968} f_s + 563 g_s)$	$x_{s+2} - 1126x_{s+1} + x_s = 0$ $y_{s+2} - 1126y_{s+1} + y_s = 0$

III. CONCLUSION

In this paper, we have presented a structured method for generating non-zero and distinct integer solutions to Pell's equation by incorporating the arithmetic properties of Wilson's primes. By applying Brahmagupta's lemma as a fundamental tool, we established recurrence relations that systematically produce infinite sequences of solutions from an initial solution pair. The factorial congruence condition characterizing Wilson's primes plays a crucial role in constructing these solution families and in strengthening the underlying algebraic structure of the equation.

The results demonstrate that the classical theory of Pell's equation can be significantly enriched through the integration of special prime classes such as Wilson primes. This interplay between prime number theory and quadratic Diophantine equations not only deepens the understanding of solution structures but also opens new directions for further research in the study of Pell-type equations and related number-theoretic problems.

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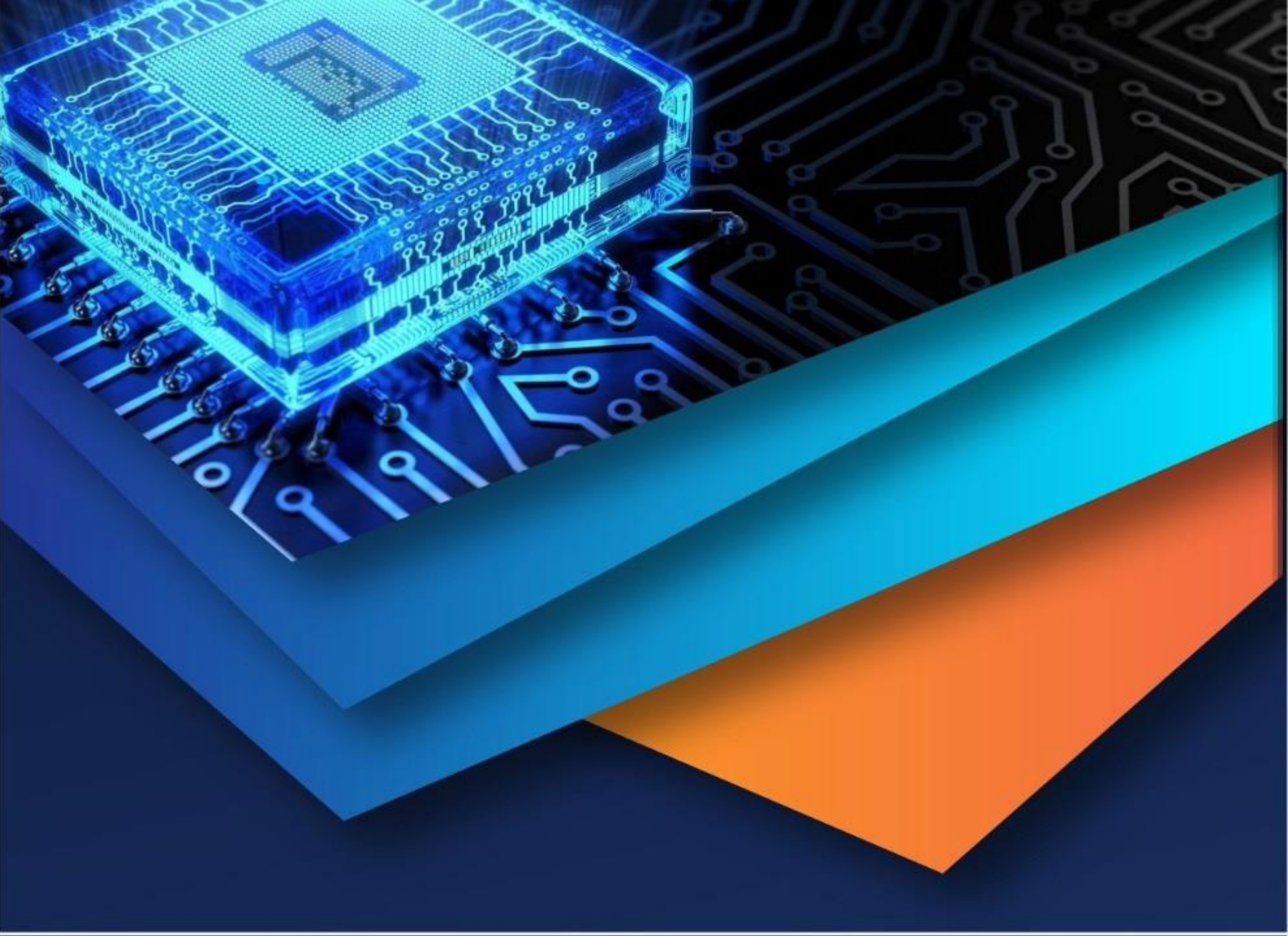

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