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The Search for Non-Negative Integer Solutions to Exponential Diophantine Equation $(3\lambda^2+5)^x+(6\lambda^2+11)^y=\mu^2$

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Abstract: In this manuscript, an exponential Diophantine equation $(3\lambda^2+5)^x+(6\lambda^2+11)^y=\mu^2$,for some selected choices of non-negative integers and $\lambda,\mu\in Z$ is scrutinized for all the amalgamation of $x+y=1,2,3$ and proved that the essentialsolutions are $(x,y,\lambda,\mu)=\{(1,0,1,3),(1,0,\tilde{\mu}_n+3\tilde{\lambda}_n,3\tilde{\mu}_n+3\tilde{\lambda}_n),(0,1,2,6),(0,1,2\widehat{\mu_n}+6\widehat{\lambda_n},6\widehat{\mu_n}+12\widehat{\lambda_n}),(1,1,1,5),(2,1,t,3\lambda^2+6)\}$ where $\tilde{\mu}_n,\tilde{\lambda}_n,\widehat{\mu_n},\widehat{\lambda_n},n\geq 0$ aregeneral solutions of some peculiar Pell equations $\mu^2=D\lambda^2+6$ where $D=3,6$ and $t\in Z$.

Keywords: Exponential Diophantine equation, Pell equation, Integer solutions.

I. INTRODUCTION

Number theory, renowned as the "Queen of Mathematics," plays a pivotal role in advancing mathematical knowledge. The study of Diophantine equations has been a central theme throughout the history of Mathematics [1,2].In [4],Suvarnamani established that the Diophantine equation $p^x+(p+1)^y=z^2$, admits a unique non-negative integer solution, namely $(p,x,y,z)=(3,1,0,2)$, when an odd prime p.In [10], Dokchan and Pakapongpun showed thatwhen p and $p+20$ are both prime numbers, the Diophantine equation$p^x+(p+20)^y=z^2$ has no positive integer solutions for (x,y,z) .In [11], authors demonstrated that the solutions of the Diophantine equation $(p+4n)^x+(p)^y=z^2$where $p,p+4n$ are prime numbers such that $p\equiv 7\ (mod\ 12)$ and n is a positive integer with$n\equiv 0,1\ (mod\ 3)$. In [3,5-9], various authors have successfully applied diverse mathematical concepts to obtain integer solutions for a range of exponential Diophantine equations. In [12,13], authors proved that the Diophantine equations $p^x+(p+2q)^y=z^2$,and $a^x+(a+2)^y=z^2$ respectively, do not acknowledge integer solutions under specific constraints.

In this assessment, an exponential Diophantine equation $(3\lambda^2+5)^x+(6\lambda^2+11)^y=\mu^2$, for some $x,y\in Z^+$ and $\lambda,\mu\in Z$ is solvedfor all the combinations of $x+y=1,2,3$and showed that the crucialsolutions are $(x,y,\lambda,\mu)=\{(1,0,1,3),(1,0,\tilde{\mu}_n+3\tilde{\lambda}_n,3\tilde{\mu}_n+3\tilde{\lambda}_n),(0,1,2,6),(0,1,2\widehat{\mu_n}+6\widehat{\lambda_n},6\widehat{\mu_n}+12\widehat{\lambda_n}),(1,1,1,5),(2,1,t,3t^2+6)\}$using the generalized solutions of the Pellian equation and fundamental concepts of Mathematics.

II. THEOREM

If x, y are restricted non-negative integers, thenanexponential Diophantine equation $(3\lambda^2+5)^x+(6\lambda^2+11)^y=\mu^2$, where $\lambda,\mu\in Z$ such that$x+y=1,2,3$ hasan enormouslylarge number of solutions $(x,y,\lambda,\mu)=\{(1,0,1,3),(1,0,\tilde{\mu}_n+3\tilde{\lambda}_n,3\tilde{\mu}_n+3\tilde{\lambda}_n),(0,1,2,6),(0,1,2\widehat{\mu_n}+6\widehat{\lambda_n},6\widehat{\mu_n}+12\widehat{\lambda_n}),(1,1,1,5),(2,1,t,3t^2+6)\}$ where $(\tilde{\mu}_n,\tilde{\lambda}_n)$ and $(\widehat{\mu_n},\widehat{\lambda_n})$, $n\geq 0$ are general solutionsto the Pell equations$\mu^2=D\lambda^2+6$where$D=3,6$and $t\in Z$.

Proof

The exponential Diophantine equation to be focussed is

$(3\lambda^2+5)^x+(6\lambda^2+11)^y=\mu^2,\ \ \lambda,\mu\in Z$ (1)

where x, y are restricted to be non-negative integers.

All the possible scenarios for $x+y=1,2,3$are enumerated below.

Case 1 :$x+y=1$

Subcase1(i): Postulate$x=1,y=0$

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Utilizingthese speculations in(1), it is renewed into

$$\mu^2 = 3\lambda^2 + 6 \quad (2)$$

which is initially fulfilled by $\mu_0 = 3, \lambda_0 = 1$

Subsequently, the above-selected options of λ and μenlist the solutions to (1) as $(x, y, \lambda, \mu) = (1,0,1,3)$

By utilizing the lowest solutions $\tilde{\mu}_0 = 2, \tilde{\lambda}_0 = 1$ of the ensuingPellian equation

$\mu^2 = 3\lambda^2 + 1$, itsall-encompassing solutionsare prearranged by

$$\tilde{\mu}_n = \frac{(2+\sqrt{3})^{n+1}+(2-\sqrt{3})^{n+1}}{2} \quad (3)$$

$$\tilde{\lambda}_n = \frac{(2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1}}{2\sqrt{3}}, n = 0,1,2\ldots \quad (4)$$

Consequently, all chances of integer solutions to (2) can be derived by

$$\mu_{n+1} = \mu_0\tilde{\mu}_n + 3\lambda_0\tilde{\lambda}_n = 3\tilde{\mu}_n + 3\tilde{\lambda}_n \quad (5)$$

$$\lambda_{n+1} = \lambda_0\tilde{\mu}_n + \mu_0\tilde{\lambda}_n = \tilde{\mu}_n + 3\tilde{\lambda}_n \quad (6)$$

Reinstate with$n = 0$, it follows from (5) and (6) entailed that$\mu_1 = 9$ and $\lambda_1 = 5$.

As a result, equation (1) simplifies to $80^x + 161^y = \mu^2$, and upon examining the potential values of x and y, delivers a solution $(x, y, \lambda, \mu) = (1,0,5,9)$.

Continuing in this manner, different values of λ and μ emerge for $n \in N$, encompassing all natural numbers, can be exemplified from(5) and (6). Leveraging these values, equation (1) yields distinct solutions, which are summarized in table 1.

Table 1

n	$\tilde{\mu}_n$	$\tilde{\lambda}_n$	Reduced form of (1)	$(x, y, \lambda, \mu) = (1,0, \tilde{\mu}_n + 3\tilde{\lambda}_n, 3\tilde{\mu}_n + 3\tilde{\lambda}_n)$	LHS of(1)
0	2	1	$80^x + 161^y = \mu^2$	(1,0,5,9)	9^2
1	7	4	$1088^x + 2177^y = \mu^2$	(1,0,19,33)	33^2
2	26	15	$15128^x + 30257^y = \mu^2$	(1,0,71,123)	123^2
3	97	56	$210680^x + 421361^y = \mu^2$	(1,0,265,459)	459^2
4	362	209	$2934368^x + 5868737^y = \mu^2$	(1,0,989,1713)	1713^2
5	1351	780	$40870448^x + 81740897^y = \mu^2$	(1,0,3691,6393)	6393^2

Subcase2(ii): Grant$x = 0, y = 1$

These two inclinations of x and y constricted (1) to the equation engagingλ and μ as

$$\mu^2 = 6\lambda^2 + 12 \quad (7)$$

The exceptionally smallest roots of (7) areassured by $\mu_0 = 6, \lambda_0 = 2$.

Consequently, one of the solutions to (1) is noted by $(x, y, \lambda, \mu) = (0,1,2,6)$.

The other capable roots of (7) are positioned through the equivalent universal equation

$$\mu^2 = 6\lambda^2 + 1 \quad (8)$$

which is gratified by $\widehat{\mu_0} = 5, \widehat{\lambda_0} = 2$.

The sequence of solutions to (8)are generalized by

$$\widehat{\mu_n} = \frac{(5+2\sqrt{6})^{n+1} + (5-2\sqrt{6})^{n+1}}{2} \quad (9)$$

$$\widehat{\lambda_n} = \frac{(5+2\sqrt{6})^{n+1} - (5-2\sqrt{6})^{n+1}}{2\sqrt{6}}, n = 0,1,2\ldots \quad (10)$$

Thus, the generalized solutions to (7) are monitored by

$$\mu_{n+1} = 6\widehat{\mu_n} + 12\widehat{\lambda_n} \quad (11)$$

$$\lambda_{n+1} = 2\widehat{\mu_n} + 6\widehat{\lambda_n} \quad (12)$$

Analysis of the cases $n = 0$, equations (11) and (12), yields the values $\mu_1 = 54$ and $\lambda_1 = 22$.

Therefore,(1) be abridged to$1457^x + 2915^y = \mu^2$andthe approved options of x and y bounces a solution$(x, y, \lambda, \mu) = (0,1,22,54)$.

Based on variousvalues of λ for $n \in N$, equation (1) can be reduced intounlikeequations whose solutions arerevealed in table 2 below.

Table 2

n	$\widehat{\mu_n}$	$\widehat{\lambda_n}$	Reduced form of (1)	$(x, y, \lambda, \mu) = (0,1,2\widehat{\mu_n} + 6\widehat{\lambda_n}, 6\widehat{\mu_n} + 12\widehat{\lambda_n})$	LHS of (1)
0	5	2	$1457^x + 2915^y = \mu^2$	(0,1,22,54)	54^2
1	49	20	$142577^x + 285155^y = \mu^2$	(0,1,218,534)	534^2
2	485	198	$13970897^x + 27941795^y = \mu^2$	(0,1,2158,5286)	5286^2
3	4801	1960	$1369005137^x + 2738010275^y = \mu^2$	(0,1,21363,52326)	52326^2
4	47525	19402	$134148532337^x + 268297064675^y = \mu^2$	(0,1,211462,517974)	517974^2
5	470449	192060	$13145187163697^x + 26290374327395^y = \mu^2$	(0,1,2093258,5127414)	5127414^2

Case3:$x + y = 2$
Subcase 3(i):Set$x = 2, y = 0$
An effect of these assortmentsis trimming downto (1) as afourth-degree equationincorporated with two parameters λ and μas

$$(3\lambda^2 + 5)^2 + 1 = \mu^2 (13)$$

In view of (13),the fact that the square of any integer plus one is not at all same to any other square integer.
Hence, it is conspicuous that(1) does not possess any solution.
Subcase3(ii):Propose $x = 1, y = 1$
Manipulation of these alternativeslessensto (1) as a second-degree equation withtwo variablesas

$$9\lambda^2 + 16 = \mu^2 (14)$$

The probable choice of $\lambda = 1$ in (14) offered the values of μ as $\mu = 5$andthere is noother feasiblesolution for any other selection of λ.
Thus, the guaranteed solution of (1) is $(x, y, \lambda, \mu) = (1,1,1,5)$.
Subcase3(iii):Set $x = 0, y = 2$
Insinuations of these preferences shrink (1) as

$$1 + (6\lambda^2 + 11)^2 = \mu^2 (15)$$

Conforming to the clarification established in the subcase3(i), the tribute produced in (15) does not hold good for integer choices of λ and μ.
As a consequence,(1) does not contributeto any integer solution.
Case 4:$x + y = 3$
Subcase4(i):Offer$x = 3, y = 0$
These assumptions streamlined (1) to the sixth-degree equation with two variable quantities as stated below

$$(3\lambda^2 + 5)^3 + 1 = \mu^2 (16)$$

The overhead equation can be customized as

$$\alpha^3 + 1 = \mu^2 (17)$$

where $\alpha = 3\lambda^2 + 5 (18)$
It tracks from the equation(17), the equality does not hold good for any value of αexcept 2.
Surrogate the value of α in(18) ascertained that

$$3\lambda^2 + 5 = 2 (19)$$

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From (19), it is penetrablefor any integer value of λ, the expression $3\lambda^2 + 5$ is not equal to 2.
For this reason, (16) is impossible for integer choices and does not yield a desired solution to (1).
Subcase4(ii):Admit$x = 2,\ y = 1$
These preferences of x and y depart (1) to the equation associated withλ and μ as

$$(3\lambda^2 + 6)^2 = \mu^2$$

$\Rightarrow \mu = 3\lambda^2 + 6 (20)$

It is crystal clear that(20) is satisfied for all $\lambda \in Z$, the set of integers. For our convenience, choose $\lambda = t, t \in Z$,then $\mu = 3t^2 + 6$.Subsequently, the Diophantine equation (1)has a solution$(x, y, \lambda, \mu) = (2, 1, t, 3t^2 + 6),\ t \in Z$.
Subcase4(iii):Assume$x = 1,\ y = 2$
The aboveselections of x and y squeeze(1) to the equation as

$$36\lambda^4 + 135\lambda^2 + 126 = \mu^2 (21)$$

Redraft(21)in the factors form shown below

$(12k + 15)(12k - 15) = (12\mu + 12)(12\mu - 12)$where $k = 6\lambda^2 + \frac{135}{12}$ (22)

The equivalent ratio of (22)can be written as

$$\frac{(12k + 15)}{(12\mu - 12)} = \frac{(12\mu + 12)}{(12k - 15)} = \frac{a}{b},\quad b \neq 0. \quad (23)$$

Similarities in (23) generates the values of k and μ as

$$k = \frac{15a^2 + 24ab + 15b^2}{12(a^2 - b^2)} (24)$$

$$\mu = \frac{6a^2 + 15ab + 6b^2}{6(a^2 - b^2)} (25)$$

It is perceived from (24) that $\lambda = \frac{150b^2 + 24ab - 120a^2}{72(a^2 - b^2)} (26)$

None of the values of a and b in (25) and (26) makeμ and λ in integers.
Consequently, (1) does not provide any integer solution for integer values of λ.
Subcase4(iv):Stipulate$x = 0,\ y = 3$
Maneuveringon these replacements diminishes(1)toan equation with degree six as

$$1 + (6\lambda^2 + 11)^3 = \mu^2 (27)$$

The assertion madeearlier contradicts the explanations provided in subcase 4(i). Consequently, equation (27) is inadmissible, and as a result, equation (1) has no solution.

III. CONCLUSION

The study of findings in this paper demonstrates that there is aninfinite number of solutions to the exponential Diophantineequation $(3\square^2 + 5)^{\square} + (6\square^2 + 11)^{\square} = \mu^2$ where $\square, \mu \in \square$, for specific combinations of $\square + \square = 1, 2, 3$. In this manner, one can search for solutions to the equations behavior for $\square + \square > 3$.

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