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The Use of Some Important Integrals in Physics

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***Abstract:** The evaluation process of the two integrals $\int r^m \cos kr\, exp(-\lambda r)dr$ and $\int r^m \sin kr\, exp(-\lambda r)dr$ Is recapitulated , using elementary knowledge from complex variables subject. This is done in a general manner, thence some special cases are worked out in full. Once ,this is done we turn to the application of these important integrals in some physical problems. In this concern, positive energy bound states is a vital example.*
***Keywords:** Integral, Physics, Important , Bound State, Complex Variables.*

I. INTRODUCTION

Some important integrals ,one meets in many scientific fields ,are

$$\int r^m \cos kr \exp(-\lambda r)\, dr \text{ and } \int r^m \sin kr \exp(-\lambda r)\, dr \tag{1}$$

Where m is a positive integer and λ ,k are real. To evaluate these two integrals using the known procedures of integration will be a very difficult task. However, if we use simple information from complex variables, we can easily compute the given integrals [1].

II. EVALUTION PROCEDURE

We note that

$$\int r^m \cos kr \exp(-\lambda r)\, dr = Re(I_m(\eta)) \tag{2}$$

,

$$\int r^m \sin kr \exp(-\lambda r)\, dr = -Im(I_m(\eta)) \tag{3}$$

And

$$I_m(\eta) = \int r^m \exp(-\eta r)\, dr \tag{4}$$

With

$$\eta = \lambda + ik \tag{5}$$

Now the last integral in Equation(4) can be evaluated easily, and

$$I_m(\eta) = \frac{-\exp(-\eta r)}{\eta}[r^m + \frac{m}{\eta}r^{m-1} + \frac{m(m-1)}{\eta^2}r^{m-2} + \cdots + \frac{m!}{\eta^m}] + c \tag{6}$$

(c is a constant of integration).

From Equations (1)-(2) and Equation (4) ,and once we know the value of m, we can compute two integrals through some simple algebra.

III. DIFFERENT CASES

To illustrate the simplicity of using the above procedure, we give the following cases

Case 1. If $\eta = 0 = k$ (a trivial case) ,then $I_m = \int r^m dr = \frac{r^{m+1}}{m+1} + c_1$.

Case 2. If $k = 0$, $then\ \eta = \lambda, and$

$$I_m = \int r^m \exp(-\lambda r)\, dr = \frac{\exp(-\lambda r)}{r}\left(r^m + \frac{m}{\lambda}r^{m-1} + \cdots \frac{m!}{\lambda^m}\right) + c_2 \tag{7}$$

A result which can obtained simply by integration by parts.

Case 3. If $m = 0\ and\ \eta = \lambda + ik, in\ this\ case$

$$I_m = \frac{-\exp(-\eta r)}{\eta} = \frac{(\cos kr - i\sin kr)(-\lambda + ik)}{|\eta|^2} + c_3 \tag{8}$$

Therefore

$$\int \cos kr \exp(-\lambda r)\, dr = \frac{(-\lambda cos\ kr + k\sin kr)}{(\lambda^2 + k^2)} \tag{9}$$

And

$$\int \sin kr \exp(-\lambda r)\, dr = \frac{(-\text{k}\cos kr + \lambda \sin kr)}{(\lambda^2 + k^2)} \quad (10)$$

This result is also attainable via the process of integration by parts.

Case 4. $m = 2,\ \lambda = 0, \eta = im$; I_m is then given by

$$I_m = \frac{i\, exdp(-ikr)}{k}\left(r^2 - \frac{2i}{k}r - \frac{2}{k^2}\right) + c_4 \quad (11)$$

From which one can obtain

$$\int r^2 \cos kr\, dr = Re\, I_m = \frac{r^2 \sin kr}{k} + \frac{2r \cos kr}{k^2} + \frac{2 \sin kr}{k^3} \quad (12)$$

Similarly, one can obtain the value of $\int r^2 \sin mr\, dr$.

Case 5. In this case, we consider the improper form of the two integrals $\int_0^\infty r^m \cos kr \exp(-\lambda r)\, dr$ and $\int_0^\infty r^m \sin kr \exp(-\lambda r)\, dr$; $\lambda > 0$.

Here, we see that

$$I_m \equiv I_m^{improper} = \frac{m!}{\eta^{m+1}} = \frac{m!(\lambda - ik)^{m+1}}{(\lambda^2 + k^2)^{m+1}} \quad (13)$$

Now, for $m = 0$,we get

$$I_0^{improper} = (\lambda - ik)/(\lambda^2 + k^2) \quad (14)$$

Which leads to

$$\int_0^\infty \cos kr \exp(-\lambda r)\, dr = \lambda/(\lambda^2 + k^2) \quad (15)$$

And

$$\int_0^\infty \sin kr \exp(-\lambda r)\, dr = k/(\lambda^2 + k^2) \quad (16)$$

For $m = 1$

$$I_1^{improper} = \frac{(\lambda - ik)^2}{(\lambda^2 + k^2)^2} = \frac{(\lambda^2 - k^2 - 2i\lambda k)}{(\lambda^2 + k^2)^2} \quad (17)$$

Equation (17) leads to the two integrals

$$\int_0^\infty r \cos kr \exp(-\lambda r)\, dr = (\lambda^2 - k^2)/(\lambda^2 + k^2)^2 \quad (18)$$

and

$$\int_0^\infty r \sin kr \exp(-\lambda r)\, dr = 2\lambda k/(\lambda^2 + k^2)^2 \quad (19)$$

For $m = 2$

$$I_2^{improper} = \frac{2(\lambda^3 - 3\lambda k^2 - 3ik\lambda^2 + ik^3)}{(\lambda^2 + k^2)^3} \quad (20)$$

Therefore

$$\int_0^\infty r^2 \cos kr \exp(-\lambda r)\, dr = 2(\lambda^3 - 3\lambda k^2)/(\lambda^2 + k^2)^3 \quad (21)$$

And

$$\int_0^\infty r^2 \sin kr \exp(-\lambda r)\, dr = 2(3k\lambda^2 - k^3)/(\lambda^2 + k^2)^3 \quad (22)$$

And so on for $m = 3,4,5 \ldots etc.$

Note that λ is assumed to be positive in the above calculations.

IV. AN ILLUSTRATIVE APPLICATION

The integrals, shown above, have their importance and usefulness in many problems in physics, especially in studying positive energy bound states(PEBS) [2],[3[,[4],[5] . Below we give an example, to show how important these integrals in solving some physical problems.

The question of PEBS is fundamental in theoretical nuclear physics and the subject has been of interest quite for a long time [5]; where it is found that for a certain class of non-local potentials ,the s-wave Schrodinger equation

$$\frac{d^2\Psi}{dr^2} + \alpha^2\Psi = \int_0^\infty K(r, r')\Psi(r')\, dr' \quad (23)$$

Has a solution $\Psi(\alpha_0, r)$,such that $\Psi(\alpha_0, r) \to 0$ as $r \to \infty$; at the energy $\alpha^2 = \alpha_0^2$. Such a solution is called a positive energy bound state; a PEBS is interpreted as a zero-width resonance .The existence of PEBS or scattering phase shifts at PEBS ,for certain conditions, have been discussed and given, leading to the need for a new interpretation [5].

Now ,generalizing to cases with any l –partial wave ($l = 0,1,2,3,..$),we see that the $l - wave$ Schrodinger equation ,with a non-local seperable potential

$$K_l(r,r') = \lambda_l g_l(r) g_l(r') \tag{24}$$

, is given as

$$\left(\frac{d^2}{dr^2} + \alpha^2 - \frac{l(l+1)}{r^2}\right)\Psi_l(r) = \lambda_l g_l(r)\int_0^\infty g_l(r')\,\Psi_l(r')\,dr' \tag{25}$$

The solution of the above equation is written as $\Psi_l(\alpha\,,r) = N_0\varphi_l(\alpha\,,r)$, with φ_l satisfying the equation

$$\left(\frac{d^2}{dr^2} + \alpha^2 - \frac{l(l+1)}{r^2}\right)\varphi_l(r) = g_l(r) \tag{26}$$

With $\varphi_l(\alpha,0) = 0$ and λ_l is given by

$$\frac{1}{\lambda_l} = \int_0^\infty g_l(r)\,\varphi_l(\alpha\,,r)dr \tag{27}$$

From Equation (25) and Equation (26), $\varphi_l(\alpha\,,r)$ is determined as

$$\varphi_l(\alpha\,,r) = \int_0^\infty G_l(r,r')g_l(r')dr' \tag{28}$$

While $G_l(r,r')$ is the Green's function given by

$$G_l(r,r') = \alpha r_< r_> j_l(\alpha r_<)\eta_l(\alpha r_>) \tag{29}$$

$j_l(\alpha r)\ and\ \eta_l(\alpha r)$ are the spherical Bessel and Neumann functions respectively [6]. Note that a PEBS ,at $\alpha^2 = \alpha_l^2$,can be obtained when $\varphi(\alpha_l,r) \to 0$ as $r \to 0$.

Now, there remains to have the form factor $g_l(r)$;which we can take as

$$g_l(r) = r^{l+2}\exp(-\beta r) \tag{30}$$

Taking this form factor into account, Equation (25) can be solved to get

$$\alpha_l^2 = (2l+3)\beta^2 \tag{31}$$

And

$$\varphi_l(\alpha_l,r) = N(\beta r^{l+2} + r^{l+1})\exp(-\beta r) \tag{32}$$

Provided that

$$\lambda_l = \frac{1}{4}\cdot\frac{(l+2)(2\beta)^{2l+2}}{(l+1)(2l+3)!} \tag{33}$$

On the other hand, if we consider the s-wave Schrodinger equation in Equation (23) with the seperable kernel

$$K(r,r') = \sum_0^n \lambda_i f_i(r) f_i(r')\ ;\ \lambda_i \neq 0 \tag{34}$$

Then the solution of the last two equations at the energy $\alpha^2 = \alpha_0^2$ such that

$$\Psi(\alpha_0,r) \to 0\ as\ r \to \infty\ ;and\ \Psi(\alpha_0,0) = 0 \tag{35}$$

is known as the PEBS wave function .This solution is written as

$$\Psi(\alpha,r) = \sum_0^\infty c_i(\alpha)\varphi_i(\alpha,r)\quad ;\ c_i \neq 0 \tag{36}$$

where $\varphi_i(\alpha,r)$ satisfies the equation

$$\left(\frac{d^2}{dr^2} + \alpha^2\right)\varphi_i(\alpha,r) = f_i(r)\ ;\ \varphi_i(\alpha,r) \to 0\ as\ r \to \infty \tag{37}$$

Taking into account all of the above equations, we reach the solution for $\varphi_i(\alpha,r)$,given as

$$\varphi_i(\alpha,r) = -\frac{1}{\alpha}\int_r^\infty \sin\alpha(r-r')f_i dr' \tag{38}$$

With

$$\varphi_i(\alpha,0) = \frac{1}{\alpha}\int_0^\infty \sin\alpha r\, f_i(r)dr \equiv g_i(\alpha) \tag{39}$$

Moreover, and putting in mind the asymptotic behavior of $\Psi(\alpha,r)$,we get

$$\Psi(\alpha,r) = \frac{1}{\alpha}\sin\alpha r + \int_i^n a_i(\alpha)\{\varphi_i(\alpha,r) - g_i(\alpha)\cos\alpha r\} \tag{40}$$

from the previous calculations in this section ,It is clear how important our integrals are; details can be referred to references [3],[4],[5].

V. CONCLUSION

As we have shown in this article ;evaluating the two integrals $\int r^m \cos kr\exp(-\lambda r)\,dr$ and $\int r^m \sin kr\exp(-\lambda r)\,dr$, was a simple task. The use of simple complex variables has made such a job simple and easy; and with no need to know the sophisticated method of the calculus of residues .Moreover, the method is elaborate, quick and instructive[1].

It should also be noted that such integrals are useful in some physical problems as we have seen in the problem of searching for a PEBS. For instance and as we have seen in section 4,if we wish to compute the scattering phase shift for a particular partial wave of order l and if the form factor is $g_l = r^{l+2}\exp(-\beta r)$,which is the modified Yukawa potential, then we run into the problem of evaluating improper integrals such as the ones encountered in case 5.

The integrals discussed and computed are just sample members of a large number of useful special integrals. To mention a few of them, we present integrals of the form

$$I_m(x_0, a_0) = \int_0^{\infty} \frac{x^m dx}{1+\exp(\frac{x-x_0}{a_0})} \quad (41)$$

Such an integral is involved in computing the two- and three-parameters Fermi or Woods Saxon function, which is attained through a ceryain binomial expansion and the computation is possible because of bounded convergence and integrability [7].Moreover ,this kind of integrals are faced with in solving heavy-ion optical model potential at intermediate energies by inversion [8]

The importance of some other special integrals of mathematical physics, such as probability integrals and Fresnel integrals ,and how to compute them in a fast manner is also tackled in an efficient way [9].

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