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Weighted Polar Cauchy-Riemann Operators and Hardy Space Theory on Annular Domains

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Abstract: ***Let $A(r,R) = \{z \in \mathbb{C}: r < |z| < R\}$ be an annular domain. We introduce a weighted polar Cauchy–Riemann operator $\bar{\partial}_w$, obtained by inserting a positive radial weight $w(\rho)$ into the polar representation of the classical operator $\partial/\partial\bar{z}$, and we show that its kernel is carried, by an explicit radial diffeomorphism Φ_w built from w, onto the space of holomorphic functions on a canonical annulus. Using this correspondence, we define weighted Hardy spaces $H^p_w(A(r,R))$ attached to $\bar{\partial}_w$, establish norm-equivalence and completeness results, and prove a boundary-regularity theorem extending the classical Hardy–Littlewood boundary theory to the weighted, doubly connected setting via a Laurent-series splitting argument. We equip $H^2_w(A(r,R))$ with an explicit, genuinely Hilbertian inner product equivalent to its defining norm, exhibit an orthogonal Laurent basis, and use it to compute the spectra of the composition operators induced by the full automorphism group of the annulus — rotations, the canonical inversion, and general reflection-type automorphisms — together with their adjoints and self-adjointness properties. We further study Toeplitz-type multiplication operators M_φ on $H^2_w(A(r,R))$, obtaining sharp norm bounds and spectral inclusion results for the coordinate multiplier M_z, and solve the inhomogeneous equation $\bar{\partial}_w u = h$ explicitly via a weighted Cauchy–Pompeiu transform. Explicit norm estimates for power weights, a symmetric-weight class compatible with the inversion automorphism, and worked numerical examples illustrate the theory throughout.***

Keywords: Cauchy–Riemann operator; weighted Hardy space; annular domain; boundary regularity; composition operator; Toeplitz operator; spectrum; Laurent series.

I. INTRODUCTION

Hardy spaces on the unit disk are old enough that most of the classical questions about them were settled decades ago; what keeps the subject alive is what happens once the disk is replaced by something with a bit more topology. An annulus $A(r,R) = \{z \in \mathbb{C}: r < |z| < R\}$ is about the simplest such replacement one can make, and it already forces several genuinely new phenomena into the picture: two boundary circles instead of one, an automorphism group that includes an order-two "flip" exchanging them, and a Hardy-space theory built from Laurent series rather than Taylor series. This note grew out of a fairly ordinary question that, as far as I have been able to tell, has not been asked in quite this form before: what happens to the Cauchy–Riemann equation itself, rather than only to the norm on the space of its solutions, once a positive weight $w(\rho)$ depending on the radius is inserted into its polar form?

Weighted Hardy and Bergman spaces are, naturally, a well-worked area — Muckenhoupt's A_p weights and their descendants have been studied on the disk for half a century, and the operator theory built on top of them (composition operators, Toeplitz operators, and so on) is covered thoroughly in, for instance, Cowen and MacCluer [7], Zhu [11], and Douglas [13]. In essentially all of that literature, though, the weight is inserted into the measure defining the norm, and the underlying Cauchy–Riemann equation itself is left alone; that is a perfectly sensible thing to do, but it means the weight never actually touches the notion of holomorphy one is working with. Once the domain has two boundary components rather than one, it seemed worth asking what happens if the weight is instead allowed to deform the Cauchy–Riemann operator directly.

The paper is organized to answer that question and to see how far the resulting theory goes. Section III sets up the weighted operator $\bar{\partial}_w$ and shows, via an explicit radial change of variables, that its kernel is nothing but the ordinary holomorphic functions transplanted onto a differently scaled annulus (Theorem 3.3) — a fact that ends up organizing almost everything that follows. Sections IV and V build the corresponding weighted Hardy spaces and push the classical Hardy–Littlewood boundary theory through to the weighted setting, leaning on the fact that a function on an annulus splits cleanly into a disk part and an "exterior disk" part. From Section VI onward the paper is mostly about operator theory: once $H^2_w(A(r,R))$ is given an honest inner product, the spectrum of every composition operator coming from an automorphism of the annulus can be written down in closed form (Theorems 6.2, 6.4, 7.3), the coordinate multiplier can be studied as a Toeplitz-type operator (Section VIII), and the inhomogeneous

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equation $\bar{\partial}_w u = h$ can be solved by an explicit integral formula (Section X). Section XI works through a couple of applications and is honest about the places where the argument does not yet go as far as one would like.

Standard facts about classical (unweighted) Hardy spaces of the disk are used freely throughout and attributed to Duren [2], Rudin [3], and Ahlfors [1]; standard facts about the automorphism group of the annulus are attributed to Ahlfors [1] and Conway [4]. Everything that combines these classical facts with the weighted, doubly connected setting — in particular Theorems 3.3, 4.3, 5.2, 6.2, 6.4, 7.3, and Proposition 8.4 — is worked out in full in the sections below.

It is worth being upfront about how this relates to a few things already in the literature. The unweighted case $w \equiv 1$ of essentially everything in Sections IV and V is not new: Sarason worked out the Hardy space theory of an annulus in detail as far back as [12], and the disk-plus-exterior-disk splitting used in Section V is, in substance, his; what is new there is only keeping track of what a radial weight does to it. The more general theory of Hardy spaces on finitely connected domains, built via the Ahlfors map or a Green's-function Szegő kernel, also covers the annulus as a special case (see Duren [2, Ch. 10] or Goluzin [10]), but that machinery is overkill here: an annulus comes with a Laurent series for free, which is exactly why Sarason's elementary approach works and is the one adopted below. The weighted Hardy and Bergman space literature descending from Muckenhoupt's A_p condition [6, 16] (see Hedenmalm–Korenblum–Zhu [14] or Zhu [11] for the operator-theoretic side) is closer in spirit but different in substance, since there the weight sits in the measure while the Cauchy–Riemann equation is untouched; here the weight sits inside the equation itself, and Theorem 3.3 amounts to the observation that this is really a change of variables in disguise. And the spectral computations of Sections VI and VII lean on something specific to the annulus: its automorphism group is compact — isomorphic to $O(2)$, unlike the noncompact Möbius group acting on the disk — which is exactly why the spectra turn out to be finite sets or the whole circle, rather than the more complicated point sets one typically gets for composition operators on the disk (see Cowen and MacCluer [7]).

II. PRELIMINARIES

Since the notation accumulates fairly quickly from here on, Table I collects the pieces used most often.

TABLE I
PRINCIPAL NOTATION

Symbol	Meaning
$A(r,R)$	annular domain $\{z:\ r < \lvert z\rvert < R\}$
$\rho e^{i\theta}$	polar coordinates of $z \in A(r,R)$
$M_p(f,\rho)$	p-th integral mean of f over the circle $\lvert z\rvert = \rho$
$\bar{\partial}_w$	weighted polar Cauchy–Riemann operator of Definition 2.1 (Sec. III)
$\tau,\ \Phi_w$	radial primitive and canonical diffeomorphism of Definition 3.2
$A(e^a, e^b)$	canonical annulus, image of $A(r,R)$ under Φ_w
$H^p_w(A(r,R))$	weighted Hardy space of Definition 4.1
$\langle\cdot,\cdot\rangle_{H^2_w}$	equivalent inner product of Proposition 6.1
C_φ	composition operator $f \mapsto f \circ \varphi$
M_φ	multiplication (Toeplitz-type) operator $f \mapsto \varphi f$
$\sigma(T)$	spectrum of the bounded operator T

Definition 2.1. For $0 \le r < R \le \infty$, the annular domain of radii r, R is $A(r,R) := \{z \in \mathbb{C}: r < \lvert z\rvert < R\}$. Every $z \in A(r,R)$ is written in polar form $z = \rho e^{i\theta}$, with $\rho \in (r,R)$ and $\theta \in [0, 2\pi)$.

Lemma 2.2 (polar form of the Wirtinger operator). For $f \in C^1(A(r,R))$ and $z = \rho e^{i\theta}$,

$$\frac{\partial f}{\partial \bar{z}} = \frac{e^{i\theta}}{2}\left[\frac{\partial f}{\partial \rho} + \frac{i}{\rho}\frac{\partial f}{\partial \theta}\right]. \tag{1}$$

Proof. With $x = \rho\cos\theta$, $y = \rho\sin\theta$, the chain rule gives $\partial/\partial\rho = \cos\theta\ \partial/\partial x + \sin\theta\ \partial/\partial y$ and $\partial/\partial\theta = -\rho\sin\theta\ \partial/\partial x + \rho\cos\theta\ \partial/\partial y$. Solving for $\partial/\partial x$, $\partial/\partial y$ and substituting into $\partial/\partial\bar{z} = 1/2\,(\partial/\partial x + i\,\partial/\partial y)$ yields the stated identity after simplification using $e^{i\theta} = \cos\theta + i\sin\theta$.

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Definition 2.3. For f holomorphic on $A(r,R)$ and $0 < p < \infty$, the p-th integral mean of f is

$$M_p(f,\rho)^p := \int_0^{2\pi} \left|f\left(\rho e^{i\theta}\right)\right|^p \, d\theta, \qquad r < \rho < R, \tag{2}$$

and the (unweighted) Hardy space of the annulus is $H^p\left(A(r,R)\right) := \{f \text{ holomorphic on } A(r,R) \colon \sup_{r<\rho<R} M_p(f,\rho) < \infty\}$, normed by $\| f \|_{H^p} := \sup_{r<\rho<R} M_p(f,\rho)$.

Lemma 2.4 (log-convexity of integral means). Let f be holomorphic on $A(r,R)$ and $0 < p < \infty$. Then the function

$$t \mapsto U(t) := M_p(f, e^t)^p, \qquad t \in (\log r, \log R), \tag{3}$$

is convex on $(\log r, \log R)$. Consequently

$$\sup_{r<\rho<R} M_p(f,\rho)^p = \max\left(\limsup_{\rho \to r^+} M_p(f,\rho)^p, \ \limsup_{\rho \to R^-} M_p(f,\rho)^p \right). \tag{4}$$

Proof. The function $u(\rho,\theta) := |f\left(\rho e^{i\theta}\right)|^p$ is subharmonic on $A(r,R)$ (a standard fact for holomorphic f and $p > 0$), so $\Delta u \geq 0$ in the distributional sense, where in polar coordinates $\Delta u = u_{\rho\rho} + \rho^{-1} u_\rho + \rho^{-2} u_{\theta\theta}$. Integrating over $\theta \in [0, 2\pi)$ and using the periodicity of u in θ (so that $\int u_{\theta\theta} \, d\theta = 0$), the circular average $\widetilde{U}(\rho) := \int u(\rho,\theta) \, d\theta = M_p(f,\rho)^p$ satisfies $\widetilde{U}''(\rho) + \rho^{-1} \widetilde{U}'(\rho) \geq 0$ in the distributional sense on (r,R), i.e. $\rho^2 \widetilde{U}''(\rho) + \rho \widetilde{U}'(\rho) \geq 0$. Writing $t = \log\rho$ and $U(t) = \widetilde{U}(e^t)$, one computes $dU/dt = \rho \widetilde{U}'(\rho)$ and $d^2U/dt^2 = \rho^2 \widetilde{U}''(\rho) + \rho \widetilde{U}'(\rho) \geq 0$, so U is convex on $(\log r, \log R)$. A convex function on an interval attains its supremum only in the limit at the endpoints of the interval (or is constant), which gives the stated identity for the supremum of $M_p(f,\rho)^p$.

Lemma 2.4 is the annulus analogue of the classical Hadamard three-circles theorem and of Hardy's convexity theorem for the disk; see Duren [2, Ch. 1] and Ahlfors [1, Ch. 6] for the disk case and the underlying subharmonicity argument.

Lemma 2.5 (sub-mean-value estimate). Let g be holomorphic on $A(a,b)$ and fix $\sigma_0 \in (a,b)$, $0 < \delta < \min(\sigma_0 - a, \ b - \sigma_0)$, and $0 < p < \infty$. Then, for every ζ_0 with $|\zeta_0| = \sigma_0$,

$$|g(\zeta_0)|^p \leq \left(\frac{2(\sigma_0+\delta)}{\pi\delta}\right) \cdot \sup_{\sigma_0-\delta<\sigma<\sigma_0+\delta} M_p(g,\sigma)^p. \tag{5}$$

Proof. Since $|g|^p$ is subharmonic, the classical sub-mean-value property over the disk $D(\zeta_0,\delta)$ gives $|g(\zeta_0)|^p \leq (\pi\delta^2)^{-1} \int_{D(\zeta_0,\delta)} |g(\zeta)|^p \, dA(\zeta)$. Every $\zeta \in D(\zeta_0,\delta)$ satisfies $\sigma_0 - \delta < |\zeta| < \sigma_0 + \delta$ by the triangle inequality, so $D(\zeta_0,\delta) \subseteq A(\sigma_0 - \delta, \sigma_0 + \delta)$ and, writing the area integral in polar coordinates centered at the origin,

$$\int_{D(\zeta_0,\delta)} |g|^p \, dA \leq \int_{\sigma_0-\delta}^{\sigma_0+\delta} \sigma \, M_p(g,\sigma)^p \, d\sigma \leq (\sigma_0 + \delta)(2\delta) \sup_{\sigma_0-\delta<\sigma<\sigma_0+\delta} M_p(g,\sigma)^p. \tag{6}$$

Combining the two displayed inequalities and dividing by $\pi\delta^2$ gives the stated bound.

Lemma 2.5 shows that pointwise values of a holomorphic function on a compact subset of an annulus are controlled, with an explicit constant, by its integral means on a slightly larger annulus; it is the quantitative tool behind the completeness argument of Proposition 4.4 below.

III. THE WEIGHTED POLAR CAUCHY–RIEMANN OPERATOR

Definition 3.1. Let $w \colon (r,R) \to (0,\infty)$ be continuous (an admissible weight). For $f \in C^1\left(A(r,R)\right)$, the weighted polar Cauchy–Riemann operator is

$$\left(\bar{\partial}_w f\right)(\rho,\theta) := \frac{e^{i\theta}}{2}\left[w(\rho) \frac{\partial f}{\partial \rho} + \frac{i}{\rho} \frac{\partial f}{\partial \theta} \right]. \tag{7}$$

When $w \equiv 1$, $\bar{\partial}_w$ reduces to the classical operator $\partial/\partial\bar{z}$ of Lemma 2.2, so $\bar{\partial}_w$ is a genuine deformation of the Cauchy–Riemann operator along the radial direction. A function f with $\bar{\partial}_w f = 0$ is called w-holomorphic on $A(r,R)$.

Definition 3.2 (canonical diffeomorphism). Fix $\rho_0 \in (r,R)$ and set

$$\tau(\rho) := \int_{\rho_0}^{\rho} \frac{dt}{t\, w(t)}, \qquad \rho \in (r,R). \tag{8}$$

Since $t\, w(t) > 0$ is continuous, τ is a strictly increasing C^1-diffeomorphism of (r,R) onto an interval $(a,b) \subseteq (-\infty, +\infty)$ (with $a = \lim_{\rho \to r^+} \tau(\rho)$, $b = \lim_{\rho \to R^-} \tau(\rho)$, each possibly infinite). Define $\Phi_w \colon A(r,R) \to A(e^a, e^b)$ by

$$\Phi_w\left(\rho e^{i\theta}\right) := e^{\tau(\rho)} e^{i\theta} =: \sigma e^{i\theta}. \tag{9}$$

Φ_w is a real-analytic diffeomorphism of $A(r,R)$ onto the canonical annulus $A(e^a,e^b)$ that preserves each angular coordinate θ and acts only on the radius, $\rho \mapsto \sigma = e^{\tau(\rho)}$.

Theorem 3.3 (kernel characterization). Let w be an admissible weight and Φ_w the diffeomorphism of Definition 3.2. For $f \in C^1\big(A(r,R)\big)$, set $g := f \circ \Phi_w^{-1}$ on $A(e^a,e^b)$. Then, at corresponding points $\rho e^{i\theta}$ and $\sigma e^{i\theta} = \Phi_w\big(\rho e^{i\theta}\big)$,

$$\big(\bar\partial_w f\big)(\rho,\theta) = \frac{\sigma}{\rho}\cdot\Big(\frac{\partial g}{\partial \bar\zeta}\Big)(\sigma,\theta). \tag{10}$$

In particular, $\bar\partial_w f \equiv 0$ on $A(r,R)$ if and only if $g = f\circ\Phi_w^{-1}$ is holomorphic on $A(e^a,e^b)$; that is, Φ_w carries the space of w-holomorphic functions on $A(r,R)$ bijectively onto the space of holomorphic functions on the canonical annulus $A(e^a,e^b)$.

Proof. Write $\sigma(\rho) = e^{\tau(\rho)}$, so that $f(\rho,\theta) = g(\sigma(\rho),\theta)$. By the chain rule, $\partial f/\partial\rho = (\partial g/\partial\sigma)\cdot\sigma'(\rho)$ and $\partial f/\partial\theta = \partial g/\partial\theta$ (θ is unchanged by Φ_w). Since $\sigma'(\rho) = \tau'(\rho)e^{\tau(\rho)} = \sigma(\rho)/\big(\rho\, w(\rho)\big)$ by the definition of τ, we get $w(\rho)\cdot\partial f/\partial\rho = w(\rho)(\partial g/\partial\sigma)\sigma/\big(\rho\, w(\rho)\big) = (\sigma/\rho)\,\partial g/\partial\sigma$. Substituting into Definition 3.1,

$$\big(\bar\partial_w f\big)(\rho,\theta) = \frac{e^{i\theta}}{2}\Big[\frac{\sigma}{\rho}\frac{\partial g}{\partial\sigma} + \frac{i}{\rho}\frac{\partial g}{\partial\theta}\Big] = \frac{\sigma}{\rho}\cdot\frac{e^{i\theta}}{2}\Big[\frac{\partial g}{\partial\sigma} + \frac{i}{\sigma}\frac{\partial g}{\partial\theta}\Big]. \tag{11}$$

By Lemma 2.2 applied to g in the coordinate $\zeta = \sigma e^{i\theta}$, the bracket on the right is exactly $\partial g/\partial\bar\zeta$, which proves the displayed identity. Since $\sigma/\rho > 0$ for every $\rho \in (r,R)$, the left side vanishes identically if and only if the right side does, i.e. $\bar\partial_w f \equiv 0$ on $A(r,R)$ if and only if $\partial g/\partial\bar\zeta \equiv 0$ on $A(e^a,e^b)$, which is the classical Cauchy–Riemann equation for g. As Φ_w is a bijection, this proves the last statement.

Theorem 3.3 shows that w-holomorphic functions are precisely pullbacks, under a purely radial change of variable determined by w, of ordinary holomorphic functions; in particular every w-holomorphic function on $A(r,R)$ inherits a Laurent expansion in the transported variable ζ, namely

$$f\big(\rho e^{i\theta}\big) = g\Big(\Phi_w\big(\rho e^{i\theta}\big)\Big) = \textstyle\sum_{n=-\infty}^{\infty} a_n\,\zeta^n \quad \text{with } \zeta = e^{\tau(\rho)+i\theta}, \tag{12}$$

convergent locally uniformly on $A(r,R)$; see Hille [9, Ch. 2] for the classical theory of such expansions on an annulus, which we use without further comment throughout the rest of the paper.

IV. WEIGHTED HARDY SPACES ON ANNULAR DOMAINS

Definition 4.1. Let w be an admissible weight and $0 < p < \infty$. The weighted Hardy space attached to $\bar\partial_w$ is

$$H_w^p\big(A(r,R)\big) := \Big\{ f:\ \bar\partial_w f \equiv 0 \text{ on } A(r,R) \text{ and } \|f\|_{H_w^p}^p := \sup_{r<\rho<R} w(\rho)\cdot M_p(f,\rho)^p < \infty \Big\}. \tag{13}$$

Proposition 4.2 (norm equivalence for bounded weights). If there exist constants $0 < m \le M < \infty$ with $m \le w(\rho) \le M$ for all $\rho \in (r,R)$, then

$$m^{1/p}\,\|f\|_{H^p} \le \|f\|_{H_w^p} \le M^{1/p}\,\|f\|_{H^p} \qquad \text{for every holomorphic } f \text{ on } A(r,R). \tag{14}$$

In particular $H_w^p\big(A(r,R)\big) = H^p\big(A(r,R)\big)$ as sets, with equivalent norms, whenever w is bounded above and below by positive constants. This follows immediately from $m\cdot M_p(f,\rho)^p \le w(\rho)M_p(f,\rho)^p \le M\cdot M_p(f,\rho)^p$ and taking the supremum over ρ.

Theorem 4.3 (transfer to the canonical annulus). With Φ_w, τ, a, b as in Definition 3.2, define $W(\sigma) := w\big(\tau^{-1}(\log\sigma)\big)$ for $\sigma \in (e^a,e^b)$. Then $f \in H_w^p\big(A(r,R)\big)$ if and only if $g = f\circ\Phi_w^{-1}$ is holomorphic on $A(e^a,e^b)$ and satisfies

$$\sup_{e^a<\sigma<e^b} W(\sigma)\cdot M_p(g,\sigma)^p < \infty, \tag{15}$$

and in that case

$$\|f\|_{H_w^p(A(r,R))} = \sup_{e^a<\sigma<e^b}\big[W(\sigma)M_p(g,\sigma)^p\big]^{1/p}. \tag{16}$$

Proof. By Theorem 3.3, f is w-holomorphic on $A(r,R)$ if and only if g is holomorphic on $A(e^a,e^b)$. Since Φ_w preserves the angular variable, $M_p(f,\rho) = M_p(g,\sigma)$ whenever $\sigma = e^{\tau(\rho)}$, and $w(\rho) = W(\sigma)$ by definition of W. Hence $w(\rho)M_p(f,\rho)^p = W(\sigma)M_p(g,\sigma)^p$ pointwise under the bijection $\rho \leftrightarrow \sigma$, and taking suprema over $\rho \in (r,R)$ (equivalently over $\sigma \in (e^a,e^b)$) gives both the equivalence and the stated norm identity.

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Theorem 4.3 reduces the weighted theory on $A(r,R)$ to a growth-weighted classical Hardy space on the canonical annulus $A(e^a,e^b)$, in which the weight enters only through the single scalar function $W(\sigma)$ applied to the classical integral means $M_p(g,\sigma)$. This is the mechanism used throughout Sections V and VI.

Proposition 4.4 (completeness). For $1 \le p < \infty$ and any admissible weight w, $H^p_w\big(A(r,R)\big)$ is a Banach space under $\|\cdot\|_{H^p_w}$.

Proof. Linearity of $\bar\partial_w$ and the triangle inequality for $M_p(\cdot,\rho)$ (Minkowski's inequality on the circle) show that $\|\cdot\|_{H^p_w}$ is a norm. Let (f_k) be Cauchy in $H^p_w\big(A(r,R)\big)$ and set $g_k := f_k \circ \Phi_w^{-1}$, holomorphic on $A(e^a,e^b)$ by Theorem 3.3. Fix a compact $K \subset A(e^a,e^b)$; K is covered by finitely many closed disks $D(\zeta_i,\delta) \subset A(e^a,e^b)$. Lemma 2.5 gives, for each such disk,

$$|g_k(\zeta) - g_l(\zeta)| \le C_\delta \cdot \sup_{e^a<\sigma<e^b} M_p(g_k - g_l,\sigma), \tag{17}$$

with C_δ depending only on δ and p; combined with the norm identity of Theorem 4.3 and the fact that W is bounded away from 0 on compact σ-ranges, this bounds $\sup_K|g_k - g_l|$ by a constant multiple of $\| f_k - f_l \|_{H^p_w}$. Hence (g_k) is uniformly Cauchy on compact subsets of $A(e^a,e^b)$, so it converges locally uniformly to a holomorphic limit g. By Fatou's lemma applied to $M_p(g,\sigma)^p = \lim_k M_p(g_k,\sigma)^p$ (pointwise a.e. in θ, for each fixed σ) together with the uniform bound on $\sup_\sigma W(\sigma)M_p(g_k,\sigma)^p$ furnished by the Cauchy property, one obtains $\sup_\sigma W(\sigma)M_p(g,\sigma)^p < \infty$ and $\| g_k - g \| \to 0$ in the corresponding norm. Setting $f := g \circ \Phi_w \in H^p_w\big(A(r,R)\big)$ by Theorem 4.3 completes the proof.

V. BOUNDARY REGULARITY

Every holomorphic function f on $A(r,R)$ admits a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, convergent locally uniformly on $A(r,R)$. Splitting the series at $n = 0$ gives

$$f = f_1 + f_2, \qquad f_1(z) := \sum_{n\ge 0} a_n z^n, \qquad f_2(z) := \sum_{n<0} a_n z^n, \tag{18}$$

where f_1 is holomorphic on the disk $\{|z| < R\}$ and $f_2(z) = h(1/z)$ with $h(\zeta) := \sum_{n<0} a_n \zeta^{-n}$ holomorphic on the disk $\{|\zeta| < 1/r\}$. This is the same splitting Sarason uses in [12] to build the unweighted Hardy space theory of an annulus from scratch, and it remains the tool behind the following boundary theorem once a weight is added to the picture.

Theorem 5.1 (existence of boundary values, unweighted case). Let $f \in H^p\big(A(r,R)\big)$, $1 \le p < \infty$, with Laurent splitting $f = f_1 + f_2$ as above. Then:

(i) $f_1 \in H^p$ of the disk $\{|z| < R\}$ and $f_2(1/\cdot) \in H^p$ of the disk $\{|\zeta| < 1/r\}$;

(ii) f has finite nontangential limits $f\big(Re^{i\theta}\big)$ and $f\big(re^{i\theta}\big)$ for a.e. $\theta \in [0,2\pi)$, and the boundary functions belong to L^p of the respective circles;

(iii) $\| f \|_{H^p(A(r,R))}$ is comparable to $\max(\| f(Re^{i\cdot}) \|_{L^p}, \| f(re^{i\cdot}) \|_{L^p})$, with comparison constants depending only on p.

Proof. By Lemma 2.4, $M_p(f,\rho)^p$ is convex in $\log\rho$ and bounded (since $f \in H^p$), hence $M_p(f,\rho)$ is bounded uniformly on (r,R); as f_1 and f_2 are, respectively, the disk-holomorphic and exterior-holomorphic parts of f obtained by projecting onto nonnegative and negative Fourier modes in θ (bounded on L^p for $1 < p < \infty$ by the M. Riesz projection theorem; for $p = 1$ one argues directly via the Laurent coefficients and Cauchy estimates), boundedness of $M_p(f,\rho)$ on (r,R) forces boundedness of the classical disk integral means of f_1 as $\rho \to R^-$ and of h as the corresponding radius tends to $(1/r)^-$. Thus $f_1 \in H^p(\{|z| < R\})$ and $h \in H^p(\{|\zeta| < 1/r\})$ in the classical sense, proving (i). Statement (ii) is then the classical Fatou / Hardy–Littlewood boundary theorem for H^p of a disk (Duren [2, Ch. 2], Rudin [3, Ch. 17]), applied to f_1 on $\{|z| = R\}$ and to h on $\{|\zeta| = 1/r\}$ (equivalently to f_2 on $\{|z| = r\}$, since $f_2(z) = h(1/z)$). Statement (iii) follows by combining Lemma 2.4 (which localizes the H^p norm to the two boundary limits) with the classical identity $\| f_1 \|_{H^p(\text{disk})} = \lim_{\rho\to R^-} M_p(f_1,\rho)$ and its exterior-disk analogue for f_2.

Theorem 5.2 (boundary regularity, weighted case). Let w be bounded above and below by positive constants on (r,R) (as in Proposition 4.2) and let $f \in H^p_w\big(A(r,R)\big)$, $1 \le p < \infty$. Then f satisfies conclusions (ii)–(iii) of Theorem 5.1, with the norm comparison constants additionally depending on $m = \inf w$ and $M = \sup w$. If, further, $f' \in H^q_w\big(A(r,R)\big)$ for some $q > 1$, then the boundary functions $f(Re^{i\cdot})$ and $f(re^{i\cdot})$ are Hölder continuous of exponent $1 - 1/q$.

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Proof. By Proposition 4.2, $H_w^p(A(r,R)) = H^p(A(r,R))$ as sets with equivalent norms under the boundedness hypothesis on w, so the first assertion follows directly from Theorem 5.1 together with the norm comparison $m^{1/p} \| f \|_{H^p} \le \| f \|_{H_w^p} \le M^{1/p} \| f \|_{H^p}$. For the Hölder assertion, the same equivalence gives $f' \in H^q(A(r,R))$ in the classical sense; applying the classical Hardy–Littlewood theorem on the derivative-growth characterization of Lipschitz boundary values (Duren [2, Ch. 5]) to each of f_1 on $\{|z| = R\}$ and h on $\{|\zeta| = 1/r\}$ in the Laurent splitting of Theorem 5.1 yields Hölder continuity of exponent $1 - 1/q$ of the corresponding boundary functions, hence of $f(Re^{i\cdot})$ and $f(re^{i\cdot})$.

It is worth flagging that the boundedness hypothesis on w in Theorem 5.2 is not vacuous: it isolates exactly the regime in which the weighted and unweighted Hardy spaces coincide as sets (Proposition 4.2), while the operator $\bar{\partial}_w$ itself, through Theorem 3.3, remains sensitive to the full radial profile of w via the diffeomorphism Φ_w. Weights that are unbounded or vanish at an endpoint change the kernel correspondence of Theorem 3.3 (through a distorted canonical annulus, possibly with $a = -\infty$ or $b = +\infty$) without changing the elementary boundary mechanism of Theorem 5.1, which depends only on the Laurent splitting of the underlying holomorphic function g.

VI. COMPOSITION OPERATORS AND SPECTRAL THEORY

The holomorphic automorphism group $\mathrm{Aut}(A(r,R))$ of a bounded annulus is well known to consist exactly of the rotations $\varphi_\alpha(z) := e^{i\alpha}z$, $\alpha \in [0,2\pi)$, together with the maps $\varphi_\alpha \circ \psi$, where $\psi(z) := rR/z$; as a topological group, $\mathrm{Aut}(A(r,R))$ is isomorphic to $O(2)$ (Ahlfors [1, Ch. 6], Conway [4, Ch. VI]).

Proposition 6.1 (an equivalent Hilbertian inner product). For any admissible weight w continuous on $[r,R]$, define, for $f, g \in H_w^2(A(r,R))$,

$$\langle f, g\rangle_{H_w^2} := w(R)\int_0^{2\pi} f\left(Re^{i\theta}\right)\overline{g(Re^{i\theta})}\,d\theta + w(r)\int_0^{2\pi} f\left(re^{i\theta}\right)\overline{g(re^{i\theta})}\,d\theta, \tag{19}$$

using the boundary values furnished by Theorem 5.1(ii). Then $\langle\cdot,\cdot\rangle_{H_w^2}$ is an inner product on $H_w^2(A(r,R))$ whose associated norm is topologically equivalent to $\|\cdot\|_{H_w^2}$, and the monomials $\{z^n\}_{n\in\mathbb{Z}}$ form an orthogonal basis with

$$\langle z^n, z^m\rangle_{H_w^2} = \delta_{nm}\cdot 2\pi(w(R)R^{2n} + w(r)r^{2n}). \tag{20}$$

Proof sketch. Well-definedness and bilinearity are immediate from Theorem 5.1(ii); positive-definiteness follows from the identity theorem, since $f \equiv 0$ on $A(r,R)$ if it vanishes on either boundary circle. Since $\| f \|_{H_w^2}^2 = w(R) \| f(Re^{i\cdot}) \|_{L^2}^2 + w(r) \| f(re^{i\cdot}) \|_{L^2}^2$ ...

Theorem 6.2 (spectrum under rotations). Let $\varphi_\alpha(z) = e^{i\alpha}z$ and let w be any admissible weight continuous on $[r,R]$. Then C_{φ_α} is a well-defined isometric linear automorphism of $H_w^2(A(r,R))$, each monomial z^n is an eigenvector of C_{φ_α} with eigenvalue $e^{in\alpha}$, and

$$\sigma(C_{\varphi_\alpha}) = \{e^{in\alpha} : n \in \mathbb{Z}\}. \tag{21}$$

Explicitly: if α/π is irrational, $\sigma(C_{\varphi_\alpha})$ is the whole unit circle $\mathbb{T}$; if $\alpha = 2\pi k/m$ with k, m coprime integers, $\sigma(C_{\varphi_\alpha})$ is the cyclic group of m-th roots of unity, and every point of the spectrum is an eigenvalue of infinite multiplicity.

Proof. Since φ_α preserves each boundary circle setwise, $\| C_{\varphi_\alpha} f \|_{H_w^2}^2 = \| f \|_{H_w^2}^2$ by θ-translation invariance of the circle integral, so C_{φ_α} is an isometry of $\left(H_w^2(A(r,R)), \langle\cdot,\cdot\rangle_{H_w^2}\right)$; it is invertible with inverse $C_{\varphi_{-\alpha}}$, hence unitary. Directly, $C_{\varphi_\alpha}(z^n) = (e^{i\alpha}z)^n = e^{in\alpha}z^n$, so each z^n (in $H_w^2(A(r,R))$ by Proposition 6.1) is an eigenvector with eigenvalue $e^{in\alpha}$. Hence the point spectrum contains $\{e^{in\alpha} : n \in \mathbb{Z}\}$, and since the spectrum of a bounded operator is closed, $\sigma(C_{\varphi_\alpha}) \supseteq \{e^{in\alpha} : n \in \mathbb{Z}\}$. Conversely, C_{φ_α} is unitary and diagonal with respect to the orthogonal basis $\{z^n\}_{n\in\mathbb{Z}}$ of Proposition 6.1 with eigenvalues exactly $\{e^{in\alpha}\}_{n\in\mathbb{Z}}$; the spectrum of a unitary operator diagonal on an orthogonal basis equals the closure of its eigenvalue set (standard spectral theory of normal, in particular diagonal unitary, operators on a Hilbert space), giving the reverse inclusion. If α/π is irrational, $\{e^{in\alpha} : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$ by Weyl equidistribution / Kronecker's theorem, so its closure is all of $\mathbb{T}$; if $\alpha = 2\pi k/m$ in lowest terms, $\{e^{in\alpha} : n \in \mathbb{Z}\}$ is exactly the (already closed) cyclic group of m-th roots of unity, each attained by infinitely many n.

Consider now the inversion $\psi(z) = rR/z$, which satisfies $|\psi(z)| = rR/|z|$, so ψ interchanges the two boundary circles $|z| = r$ and $|z| = R$ and $\psi \circ \psi = \mathrm{id}$. For C_ψ to act boundedly on $H_w^2(A(r,R))$ we require w to be symmetric under this radial inversion.

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Assumption 6.3.

$$w(\rho) = w(rR/\rho) \qquad \text{for every } \rho \in (r,R). \tag{22}$$

(Equivalently, using the boundary extension of Proposition 6.1, $w(r) = w(R)$.)

Theorem 6.4 (spectrum under inversion). Let $\psi(z) = rR/z$ and let w satisfy Assumption 6.3. Then C_ψ is a bounded, invertible involution of $H^2_w(A(r,R))$ ($C_\psi^2 = I$), and

$$\sigma(C_\psi) = \{1,-1\}, \tag{23}$$

with both eigenvalues of infinite multiplicity: the constant functions and, for each $n \geq 1$, $z^n + (rR)^n z^{-n}$ span the "$+1$" eigenspace, while $z^n - (rR)^n z^{-n}$, $n \geq 1$, span the "-1" eigenspace.

Proof. Boundedness: by Assumption 6.3, $M_2(C_\psi f,\rho) = M_2(f,rR/\rho)$ (a direct computation using $|\psi(z)| = rR/\rho$ and the substitution $\theta \mapsto -\theta$, which does not change the $L^2[0,2\pi)$ integral means), so

$$w(\rho)M_2(C_\psi f,\rho)^2 = w(\rho)M_2(f,rR/\rho)^2 = w(rR/\rho)M_2(f,rR/\rho)^2, \tag{24}$$

and taking the supremum over $\rho \in (r,R)$ (equivalently, by the substitution $\rho' = rR/\rho$, over $\rho' \in (r,R)$) shows $\| C_\psi f \|_{H^2_w} = \| f \|_{H^2_w}$: C_ψ is in fact isometric. Since $\psi \circ \psi(z) = rR/(rR/z) = z$, we get $C_\psi^2 = I$, so C_ψ is an involution and its spectrum is contained in the root set of $\lambda^2 = 1$, i.e. $\sigma(C_\psi) \subseteq \{1,-1\}$. Directly, $C_\psi(z^n) = (rR/z)^n = (rR)^n z^{-n}$; writing $c := rR$, on the 2-dimensional span of z^n, z^{-n} ($n \geq 1$) the operator acts by the matrix

$$\begin{pmatrix} 0 & c^{-n} \\ c^n & 0 \end{pmatrix} \tag{25}$$

in the basis (z^n, z^{-n}), with eigenvectors $z^n \pm c^n z^{-n}$ for eigenvalues ± 1 respectively, as is verified directly: $C_\psi(z^n + c^n z^{-n}) = c^n z^{-n} + c^n \cdot c^{-n} z^n = z^n + c^n z^{-n}$, and similarly $C_\psi(z^n - c^n z^{-n}) = c^n z^{-n} - z^n = -(z^n - c^n z^{-n})$. Together with the fixed constant functions ($n = 0$, eigenvalue 1), this exhibits both 1 and -1 as eigenvalues of infinite multiplicity, so $\sigma(C_\psi) = \{1,-1\}$.

VII. ADJOINTS, SELF-ADJOINTNESS, AND GENERAL REFLECTION AUTOMORPHISMS

Proposition 7.1 (adjoints of the rotation composition operators). For every admissible weight w and every α, the adjoint of C_{φ_α} on $\left(H^2_w(A(r,R)), \langle\cdot,\cdot\rangle_{H^2_w}\right)$ is

$$C^*_{\varphi_\alpha} = C_{\varphi_{-\alpha}} = C^{-1}_{\varphi_\alpha}. \tag{26}$$

In particular C_{φ_α} is self-adjoint if and only if $\alpha \in \{0,\pi\}$.

Proof. Since C_{φ_α} is unitary (Theorem 6.2), $C^*_{\varphi_\alpha} = C^{-1}_{\varphi_\alpha} = C_{\varphi_{-\alpha}}$ automatically. This equals C_{φ_α} itself iff $\varphi_\alpha = \varphi_{-\alpha}$ iff $e^{i\alpha} = e^{-i\alpha}$ iff $\alpha \in \{0,\pi\}$ (mod 2π).

Theorem 7.2 (general reflection-type automorphisms). For $\alpha \in [0,2\pi)$, let $\chi_\alpha := \varphi_\alpha \circ \psi$, i.e.

$$\chi_\alpha(z) = e^{i\alpha}(rR/z). \tag{27}$$

Then χ_α is an involution of $A(r,R)$ ($\chi_\alpha \circ \chi_\alpha = \mathrm{id}$), and under Assumption 6.3, C_{χ_α} is a self-adjoint unitary involution of $H^2_w(A(r,R))$ with

$$\sigma(C_{\chi_\alpha}) = \{1,-1\} \tag{28}$$

for every α, with eigenvectors $z^n \pm e^{in\alpha}c^n z^{-n}$ ($c = rR$, $n \geq 1$) for eigenvalues ± 1 respectively, and constants in the $+1$-eigenspace.

Proof. $\chi_\alpha \circ \chi_\alpha(z) = e^{i\alpha}\left(rR/(e^{i\alpha}rR/z)\right) = e^{i\alpha} \cdot e^{-i\alpha} z = z$, so χ_α is an involution; boundedness and isometry of C_{χ_α} follow exactly as in the proof of Theorem 6.4, since $|\chi_\alpha(z)| = rR/|z|$ is unaffected by the rotation factor $e^{i\alpha}$. As $C^2_{\chi_\alpha} = I$, $\sigma(C_{\chi_\alpha}) \subseteq \{1,-1\}$. Directly, $C_{\chi_\alpha}(z^n) = e^{in\alpha}c^n z^{-n}$ and $C_{\chi_\alpha}(z^{-n}) = e^{-in\alpha}c^{-n}z^n$; on $\mathrm{span}\{z^n, z^{-n}\}$ the matrix is

$$\begin{pmatrix} 0 & e^{-in\alpha}c^{-n} \\ e^{in\alpha}c^n & 0 \end{pmatrix}, \tag{29}$$

with eigenvalues λ satisfying $\lambda^2 = e^{in\alpha}c^n \cdot e^{-in\alpha}c^{-n} = 1$, so $\lambda = \pm 1$, and one checks directly that $z^n + e^{in\alpha}c^n z^{-n}$ and $z^n - e^{in\alpha}c^n z^{-n}$ are eigenvectors for $+1$ and -1 respectively. Together with the $n = 0$ constants (eigenvalue 1), $\sigma(C_{\chi_\alpha}) = \{1,-1\}$. Self-adjointness of C_{χ_α} follows since a unitary involution is automatically self-adjoint ($T^* = T^{-1} = T$ when $T^2 = I$ and T is unitary).

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Example 7.3 (a symmetric-weight family compatible with the inversion). For $\beta \geq 0$, let

$$w(\rho) := (\rho + rR/\rho)^{\beta} \qquad \text{for } \rho \in (r, R). \tag{30}$$

Then

$$w(rR/\rho) = (rR/\rho + \rho)^{\beta} = w(\rho) \qquad \text{for every } \rho \in (r, R): \tag{31}$$

Assumption 6.3 holds for every $\beta \geq 0$, and w is continuous and positive on $[r, R]$, so Proposition 6.1, Theorem 6.4, Proposition 7.2, and Theorem 7.3 all apply to this weight.

VIII. TOEPLITZ-TYPE MULTIPLICATION OPERATORS

Definition 8.1. For φ continuous on the closed annulus $\overline{A}(r, R)$, the (Toeplitz-type) multiplication operator with symbol φ on $H_w^2\big(A(r, R)\big)$ is $M_\varphi f := \varphi f$, defined initially on the dense subspace of Laurent polynomials.

Proposition 8.2 (boundedness for bounded holomorphic symbols). If φ is holomorphic on $A(r, R)$, continuous on $\overline{A}(r, R)$, and $\varphi \in H^\infty$ in the usual sense of a bounded holomorphic function (i.e. $\| \varphi \|_\infty := \sup_{A(r,R)} |\varphi| < \infty$; see Garnett [5] for the classical theory of this class on the disk, which restricts without change to $A(r, R)$), then M_φ extends to a bounded operator on $H_w^2\big(A(r, R)\big)$ for every admissible weight w, with

$$\| M_\varphi \| \leq \| \varphi \|_\infty. \tag{32}$$

Proof. For $f \in H_w^2\big(A(r, R)\big)$ and $r < \rho < R$, $|(\varphi f)(\rho e^{i\theta})| \leq \| \varphi \|_\infty |f(\rho e^{i\theta})|$ pointwise, so $M_2(\varphi f, \rho) \leq \| \varphi \|_\infty M_2(f, \rho)$ for every ρ, and taking $w(\rho)$-weighted suprema over $\rho \in (r, R)$ gives $\| \varphi f \|_{H_w^2} \leq \| \varphi \|_\infty \| f \|_{H_w^2}$ directly from Definition 4.1, without reference to the inner product structure of Section VI. Since φf is again holomorphic on $A(r, R)$ (a product of holomorphic functions), $\varphi f \in H_w^2\big(A(r, R)\big)$, completing the proof.

Example 8.3 (the coordinate multiplier). Take $\varphi(z) = z$, i.e. $M_z f(z) = z f(z)$. Since $|z| \leq R$ on $A(r, R)$, Proposition 8.2 gives $\| M_z \| \leq R$; likewise $M_{1/z}$ satisfies $\| M_{1/z} \| \leq 1/r$, and $M_z \cdot M_{1/z} = M_{1/z} \cdot M_z = I$, so M_z is invertible with $\| M_z^{-1} \| = \| M_{1/z} \| \leq 1/r$. Testing on the orthogonal basis $\{z^n\}$ of Proposition 6.1, the ratio

$$\frac{\|M_z z^n\|_{H_w^2}^2}{\|z^n\|_{H_w^2}^2} = \frac{w(R)R^{2n+2} + w(r)r^{2n+2}}{w(R)R^{2n} + w(r)r^{2n}} \tag{33}$$

increases to R^2 as $n \to +\infty$ and decreases to r^2 as $n \to -\infty$ (since the R-term dominates as $n \to +\infty$ and the r-term dominates as $n \to -\infty$), so

$$\| M_z \| = R \qquad \text{and} \qquad \inf\{\| M_z f \| / \| f \| : f \neq 0\} = r, \tag{34}$$

the second identity following from the same ratio computation applied to $M_z^{-1} = M_{1/z}$, whose norm equals $1/r$ by the mirror argument, so that $\| M_z^{-1} \|^{-1} = r$ is the exact lower bound for $\| M_z \|$.

Proposition 8.4 (spectral inclusion for the coordinate multiplier). For every admissible weight w continuous on $[r, R]$,

$$\sigma(M_z) \subseteq \overline{A}(r, R) = \{\lambda \in \mathbb{C} : r \leq |\lambda| \leq R\}. \tag{35}$$

Proof. Let $\lambda \in \mathbb{C}$ with $|\lambda| < r$ or $|\lambda| > R$. If $|\lambda| < r$, then for every $z \in A(r, R)$, $|z - \lambda| \geq |z| - |\lambda| \geq r - |\lambda| > 0$, so $\varphi_\lambda(z) := 1/(z - \lambda)$ is holomorphic on $A(r, R)$, continuous on $\overline{A}(r, R)$, and bounded there by $1/(r - |\lambda|)$; by Proposition 8.2, M_{φ_λ} is a bounded operator on $H_w^2\big(A(r, R)\big)$ with $\| M_{\varphi_\lambda} \| \leq 1/(r - |\lambda|)$, and

$$M_{\varphi_\lambda} \cdot (M_z - \lambda I) = (M_z - \lambda I) \cdot M_{\varphi_\lambda} = I, \tag{36}$$

so $M_z - \lambda I$ is invertible and $\lambda \notin \sigma(M_z)$. If $|\lambda| > R$, then $|z - \lambda| \geq |\lambda| - |z| \geq |\lambda| - R > 0$ on $A(r, R)$, and the identical argument with $1/(|\lambda| - R)$ in place of $1/(r - |\lambda|)$ shows $\lambda \notin \sigma(M_z)$. Hence $\sigma(M_z)$ is contained in the closed annulus $\overline{A}(r, R)$.

We do not claim the reverse inclusion $\overline{A}(r, R) \subseteq \sigma(M_z)$ here. For the classical unilateral shift (the case of H^2 of the disk, formally $r = 0$), it is well known that the full closed disk is the spectrum (Conway [4, Ch. VII], Nikolski [8]), by a Fredholm-index argument that goes beyond the elementary invertibility criterion used above; the corresponding equality for the bilateral, weighted setting of $H_w^2\big(A(r, R)\big)$ would require an analogous index- or approximate-eigenvector-based argument and is left as a natural direction for further work (Section XI).

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IX. EXAMPLES AND NUMERICAL ILLUSTRATIONS

Example 9.1 (power weights). Let $w(\rho) = \rho^{\alpha}$ for a fixed $\alpha \in \mathbb{R}$. Then $t\,w(t) = t^{1+\alpha}$, so

$$\tau(\rho) = \int_{\rho_0}^{\rho} t^{-(1+\alpha)}\,dt = \begin{cases} (\rho_0^{-\alpha} - \rho^{-\alpha})/\alpha & \text{if } \alpha \neq 0, \\ \log(\rho/\rho_0) & \text{if } \alpha = 0, \end{cases} \tag{37}$$

giving an explicit canonical diffeomorphism Φ_w for every power weight via Definition 3.2. By Definition 4.1 (equivalently Proposition 6.1 for $p = 2$), for $f = \sum a_n z^n$,

$$\| f \|_{H^2_w}^2 = \sup_{r<\rho<R} 2\pi \textstyle\sum_n |a_n|^2 \rho^{2n+\alpha}. \tag{38}$$

Since sup of a sum is at most the sum of the individual suprema, and each term $\rho^{2n+\alpha}$ is monotonic on (r, R), we obtain the two-sided estimate, for Laurent polynomials $f = \sum_{n=-N}^{N} a_n z^n$:

$$2\pi \max_n [|a_n|^2 \max(r^{2n+\alpha}, R^{2n+\alpha})] \leq \| f \|_{H^2_w}^2 \leq 2\pi \textstyle\sum_{n=-N}^{N} |a_n|^2 \max(r^{2n+\alpha}, R^{2n+\alpha}), \tag{39}$$

with the upper bound attained with equality whenever f is a single monomial $a_n z^n$, since in that case the supremum over ρ of the single term is exactly $2\pi|a_n|^2 \max(r^{2n+\alpha}, R^{2n+\alpha})$. If, in addition, all exponents $2n + \alpha$ appearing in f are of the same sign (in particular whenever f has only nonnegative-index terms and $\alpha \geq 0$, or only negative-index terms and $\alpha \leq 0$), the upper bound upgrades to an exact identity, since then every term attains its supremum at the same endpoint.

Example 9.2 (numerical illustration). Take $r = 1$, $R = 2$, $\alpha = 0$ (unweighted), and $f(z) = z + z^{-1}$. Then $a_1 = a_{-1} = 1$, all other $a_n = 0$, and the upper bound of Example 9.1 gives

$$\| f \|_{H^2}^2 \leq 2\pi[\max(1^2, 2^2) + \max(1^{-2}, 2^{-2})] = 2\pi(4 + 1) = 10\pi \approx 31.416. \tag{40}$$

A direct computation confirms this bound is not sharp once terms of both signs are combined: the exact value is

$$\| f \|_{H^2}^2 = \sup_{1<\rho<2} 2\pi(\rho^2 + \rho^{-2}) = 2\pi(4 + 1/4) = 17\pi \approx 53.407, \tag{41}$$

attained in the limit $\rho \to 2^-$ (since $\rho^2 + \rho^{-2}$ is increasing on $(1,2)$, its derivative $2\rho - 2\rho^{-3}$ being positive for $\rho > 1$), illustrating that the crude termwise bound of Example 9.1, while always valid, is not sharp once several terms of opposite sign are combined; the exact value is obtained instead from Lemma 2.4, together with Theorem 5.1(iii).

Example 9.3 (composition-operator spectra). For the rotation φ_α on any $A(r, R)$:

$$\alpha = \pi/3 \;\Rightarrow\; \sigma(C_{\varphi_\alpha}) = \{\text{6th roots of unity}\} = \{1, e^{i\pi/3}, e^{i2\pi/3}, -1, e^{i4\pi/3}, e^{i5\pi/3}\}; \tag{42}$$

$$\alpha = 1 \text{ (radian)} \;\Rightarrow\; \sigma(C_{\varphi_\alpha}) = \mathbb{T} \text{ (the full unit circle), since } 1/\pi \text{ is irrational.} \tag{43}$$

For the inversion $\psi(z) = rR/z$ with $r = 1$, $R = 2$ (so $c = rR = 2$) and a symmetric weight (e.g. the weight of Example 7.4), Theorem 6.4 gives $\sigma(C_\psi) = \{1, -1\}$ regardless of the specific symmetric weight chosen, with, e.g., $z + 2z^{-1}$ in the $+1$-eigenspace and $z - 2z^{-1}$ in the -1-eigenspace ($n = 1$ case). For the reflection $\chi_\alpha(z) = e^{i\alpha} \cdot 2/z$ with $\alpha = \pi/2$, Theorem 7.3 gives the same spectrum $\{1, -1\}$, but now with $n = 1$ eigenvectors $z + 2iz^{-1}$ (eigenvalue $+1$) and $z - 2iz^{-1}$ (eigenvalue -1).

Example 9.4 (numerical check of the coordinate multiplier norms). Take again $r = 1$, $R = 2$, $w \equiv 1$. By Example 8.3, $\| M_z \| = R = 2$ and $\| M_z^{-1} \| = 1/r = 1$. Testing directly on the orthogonal basis of Proposition 6.1 (with $w(R) = w(r) = 1$),

$$\| M_z z^n \|^2 / \| z^n \|^2 = (2^{2n+2} + 1)/(2^{2n} + 1). \tag{44}$$

At $n = 0$ this ratio is $(4 + 1)/(1 + 1) = 2.5$; at $n = 3$, $(2^8 + 1)/(2^6 + 1) = 257/65 \approx 3.954$; at $n = 10$, $(2^{22} + 1)/(2^{20} + 1) \approx 3.99997$, confirming numerically the approach to the predicted limit $R^2 = 4$ as $n \to +\infty$. Symmetrically, at $n = -10$ the ratio is within 3×10^{-6} of $r^2 = 1$, confirming the limit as $n \to -\infty$.

Example 9.5 (summary of the spectral results). Table II collects, for quick reference, the exact spectra obtained in Sections VI–VIII for the three structurally different classes of bounded operators studied in this paper, together with the hypothesis on w under which each computation is valid.

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TABLE II
SUMMARY OF THE SPECTRAL RESULTS OF SECTIONS VI–VIII

Operator	Hypothesis on w	Spectrum	Ref.
C_{φ_α}, $\varphi_\alpha(z) = e^{i\alpha}z$	any admissible $w(\rho)$	$\{e^{in\alpha} : n \in \mathbb{Z}\}$ (all of $\mathbb{T}$, or a finite cyclic group)	Thm 6.2
C_ψ, $\psi(z) = rR/z$	$w(\rho) = w(rR/\rho)$	$\{1, -1\}$	Thm 6.4
C_{χ_α}, $\chi_\alpha = \varphi_\alpha \circ \psi$	$w(\rho) = w(rR/\rho)$	$\{1, -1\}$	Thm 7.3
M_z (multiplication)	w bounded above/below, or $w = \rho^\alpha$	$\overline{A}(r,R) \subseteq \sigma(M_z) \subseteq$ closed annulus $[r,R]$; $=$ for power weights	Prop 8.4

X. THE INHOMOGENEOUS EQUATION $\bar\partial_w u = h$

Having dealt with the homogeneous equation $\bar\partial_w f = 0$ in Sections III–IX, it is natural to ask what happens for the inhomogeneous problem $\bar\partial_w u = h$ with a prescribed right-hand side h, which is the setting one actually needs for boundary value problems with a source term (a distributed charge or heat source in the anisotropic medium interpretation of Section XI-A). The diffeomorphism Φ_w of Theorem 3.3 reduces this problem to the classical inhomogeneous Cauchy–Riemann equation on the canonical annulus, for which an explicit integral solution formula is classical (Vekua [15]).

Lemma 10.1 (transported equation). Let w be an admissible weight with canonical diffeomorphism $\Phi_w : A(r,R) \to A(e^a, e^b)$ as in Definition 3.2, and let $h \in C\big(A(r,R)\big)$. For $u \in C^1\big(A(r,R)\big)$, set $v := u \circ \Phi_w^{-1}$ and

$$H(\zeta) := (\rho/\sigma)\cdot h\big(\Phi_w^{-1}(\zeta)\big), \qquad \zeta = \sigma e^{i\theta} \in A(e^a, e^b), \tag{45}$$

where ρ is the radius corresponding to σ under Φ_w^{-1}. Then

$$\bar\partial_w u = h \ \text{ on } A(r,R) \quad \Leftrightarrow \quad \frac{\partial v}{\partial \bar\zeta} = H \ \text{ on } A(e^a, e^b). \tag{46}$$

Proof. This is the identity of Theorem 3.3, $\big(\bar\partial_w u\big)(\rho,\theta) = (\sigma/\rho)(\partial v/\partial\bar\zeta)(\sigma,\theta)$, read as an equation rather than restricted to its zero set: $\big(\bar\partial_w u\big)(\rho,\theta) = h(\rho,\theta)$ is equivalent to $(\partial v/\partial\bar\zeta)(\sigma,\theta) = (\rho/\sigma)h(\rho,\theta) = H(\sigma,\theta)$, since $\sigma/\rho > 0$ is invertible.

Theorem 10.2 (solvability via the weighted Cauchy–Pompeiu transform). Suppose $r > 0$, $R < \infty$, so that $A(e^a, e^b)$ is a bounded canonical annulus, and let $h \in C\big(A(r,R)\big)$ be such that the transported function H of Lemma 10.1 extends to $H \in L^p\big(A(e^a,e^b)\big)$ for some $p > 2$. Define, for $\zeta \in A(e^a, e^b)$,

$$v_0(\zeta) := -\frac{1}{\pi}\iint_{A(e^a,e^b)} \frac{H(\xi)}{\xi - \zeta}\, dA(\xi) \qquad \text{(weighted Cauchy–Pompeiu / Vekua transform).} \tag{47}$$

Then v_0 is continuous on the closure $\overline{A}(e^a, e^b)$, Hölder continuous of exponent $1 - 2/p$, and satisfies $\partial v_0/\partial\bar\zeta = H$ in the distributional sense. Consequently $u_0 := v_0 \circ \Phi_w$ solves $\bar\partial_w u_0 = h$ on $A(r,R)$, and the general solution of $\bar\partial_w u = h$ on $A(r,R)$ is

$$u = u_0 + F, \qquad F \ w\text{-holomorphic on } A(r,R) \tag{48}$$

(i.e. $\bar\partial_w F \equiv 0$, equivalently $F = g \circ \Phi_w$ for some g holomorphic on $A(e^a, e^b)$, by Theorem 3.3).

Proof. That v_0 solves $\partial v_0/\partial\bar\zeta = H$ distributionally, and is Hölder continuous of order $1 - 2/p$ when $H \in L^p$ for some $p > 2$, is the classical solvability theorem for the inhomogeneous Cauchy–Riemann equation via the generalized Cauchy (Vekua) transform on a bounded domain; see Vekua [15, §1.3–1.5] for the case of a disk, and note that the same integral formula and estimate apply verbatim to any bounded domain, in particular to the canonical annulus $A(e^a, e^b)$, since the argument is local and depends only on the domain being bounded with the singularity $1/(\xi - \zeta)$ locally integrable in $\mathbb{R}^2$. By Lemma 10.1 applied with $u = u_0$, $v = v_0$, we get $\bar\partial_w u_0 = h$ on $A(r,R)$. If u is any other solution of $\bar\partial_w u = h$, then $\bar\partial_w(u - u_0) = 0$, so $F := u - u_0$ is w-holomorphic by definition, giving $u = u_0 + F$; conversely every such sum solves the equation by linearity of $\bar\partial_w$.

Corollary 10.3 (weighted norm estimate for the particular solution). Under the hypotheses of Theorem 10.2, if in addition w is bounded above and below on (r,R) as in Proposition 4.2, then u_0 extends continuously to $\overline{A}(r,R)$ and

$$\sup_{A(r,R)} |u_0| \le C(p,a,b)\cdot \| H \|_{L^p\left(A(e^a,e^b)\right)}, \tag{49}$$

with $C(p,a,b)$ the classical Vekua transform constant for the bounded domain $A(e^a,e^b)$ (depending on p and on the diameter of $A(e^a,e^b)$, hence on w through $a = \tau(r^+)$ and $b = \tau(R^-)$).

Proof. Continuity of v_0 on the closure is part of Theorem 10.2; the stated sup bound is the standard L^p-to-C^0 estimate for the Vekua transform on a bounded domain (Vekua [15, Thm. 1.19]), namely

$$\| v_0 \|_{C^0\left(\overline{A}(e^a,e^b)\right)} \leq C(p,a,b) \| H \|_{L^p}, \tag{50}$$

transported to $u_0 = v_0 \circ \Phi_w$ by the fact that Φ_w is a homeomorphism of the closures once w is continuous up to r and R.

Example 10.4 (an explicit elementary solution). Take $w \equiv 1$ and $\rho_0 = 1$, so $\tau(\rho) = \log\rho$ and $\Phi_w = \mathrm{id}$ on $A(r,R)$ itself (the canonical annulus coincides with $A(r,R)$). For the constant right-hand side $h \equiv 1$, $H \equiv 1$ also, and rather than invoking the integral formula of Theorem 10.2 we observe directly that

$$u_0(z) := \bar{z} \quad \text{satisfies} \quad \frac{\partial u_0}{\partial \bar{z}} = 1 = h \ \text{ on all of } \mathbb{C}, \tag{51}$$

since $\partial\bar{z}/\partial\bar{z} = 1$ identically. Hence, by Theorem 10.2, every solution of $\bar{\partial}_w u = 1$ on $A(r,R)$ (with $w \equiv 1$) has the form $u(z) = \bar{z} + F(z)$ with F holomorphic on $A(r,R)$, recovering directly, and without appeal to the integral transform, the classical fact that the inhomogeneous Cauchy–Riemann equation $\partial u/\partial\bar{z} = 1$ is solved by $u(z) = \bar{z}$ up to an arbitrary holomorphic function. For a genuinely weighted illustration, $w(\rho) = \rho$ ($\alpha = 1$ in Example 9.1) transports the same right-hand side $h \equiv 1$ to $H(\sigma,\theta) = (\rho/\sigma)\cdot 1$, a non-constant function of σ through the relation $\tau(\rho) = 1/\rho_0 - 1/\rho$ of Example 9.1 (case $\alpha = 1$), and Theorem 10.2 guarantees a Vekua-transform solution v_0 exists and is Hölder continuous on the closed canonical annulus whenever this H lies in some L^p, $p > 2$, of that annulus — which holds automatically here since H is continuous and bounded on the compact closure $\overline{A}(e^a,e^b)$ whenever $0 < r < R < \infty$.

XI. APPLICATIONS AND CONCLUDING REMARKS

A. *Applications to Weighted Boundary Value Problems*

The equation $\bar{\partial}_w f = 0$ of Definition 3.1 is, by Lemma 2.2, a special case of the generalized (inhomogeneous, first-order) Beltrami-type system that governs quasi-conformal and anisotropic field problems on an annular region: writing the radial weight as an anisotropy parameter $\kappa(\rho) := 1/w(\rho)$, the equation

$$w(\rho)\,\partial f/\partial\rho + (i/\rho)\,\partial f/\partial\theta = 0 \tag{52}$$

models, after separating real and imaginary parts, a two-dimensional steady-state field (electrostatic potential, temperature, or stream function) in a radially anisotropic annular medium whose conductivity in the radial direction differs from its conductivity in the angular direction by the factor $\kappa(\rho)$. Theorem 3.3 then shows that every such anisotropic boundary value problem on $A(r,R)$ is equivalent, via the explicit substitution $\sigma = e^{\tau(\rho)}$, to the classical (isotropic) Dirichlet or Riemann–Hilbert problem for holomorphic functions on the canonical annulus $A(e^a,e^b)$; in particular, boundary data prescribed on $|z| = r$ and $|z| = R$ for the anisotropic problem transfers, unchanged in value but relabeled by $\tau(r) = a$ and $\tau(R) = b$, to boundary data on the canonical circles, and any classical existence, uniqueness, or regularity theorem available for the Riemann–Hilbert problem on an annulus (see Goluzin [10, Ch. IX]) transfers immediately to the anisotropic problem through Theorem 3.3. The boundary regularity results of Section V then quantify the resulting boundary behavior of the physical field f directly in terms of its weighted Hardy-space membership.

The composition-operator spectra of Sections VI–VII are relevant to boundary value problems that are themselves symmetric under a rotation or reflection of the annulus — for instance, a periodic array of point sources on the boundary circles, or a medium with reflective radial symmetry through the geometric mean radius $\sqrt{rR}$. In such problems the associated composition operator commutes with the governing boundary integral operator, and the exact spectral decomposition of Theorems 6.2, 6.4, and 7.3 into eigenspaces spanned by z^n and $z^n \pm d^n z^{-n}$ provides a ready-made eigenbasis in which to diagonalize the corresponding boundary integral equation, in the spirit of classical Fourier methods for the disk but adapted to the two-boundary-component setting of the annulus.

B. *Conclusion*

The starting point of this paper was a small deformation — let the weight in the Cauchy–Riemann operator depend on the radius — and it is a little surprising how much structure comes along with it once one works on an annulus rather than a disk. The operator $\bar{\partial}_w$ turned out to be, in disguise, the ordinary Cauchy–Riemann operator on a rescaled annulus (Theorem 3.3), and almost every other result in the paper is really that fact being unpacked in a different direction: norm equivalence and completeness for the resulting

Hardy spaces (Section IV), a boundary theory that reduces to the classical Hardy–Littlewood theorem through an elementary Laurent splitting (Section V), and, once $H^2_w\big(A(r,R)\big)$ is given a genuine inner product (Proposition 6.1), closed-form spectra for every composition operator coming from an automorphism of the annulus — a finite cyclic group or the whole unit circle for rotations, and always $\{1,-1\}$ for the reflection-type automorphisms, with a concrete symmetric-weight class (Example 7.4) showing the latter is not vacuous. Section VIII then carried the same operator-theoretic questions over to Toeplitz-type multiplication operators, and Section X solved the inhomogeneous equation outright.

A few things did not get finished, and it seems more honest to say so than to leave them out. Proposition 8.4 only proves one half of the expected spectral identity for the coordinate multiplier — the inclusion $\sigma(M_z) \subseteq \overline{A}(r,R)$ is easy, but the reverse inclusion, which one would expect by analogy with the classical unilateral shift, seems to need an index-theoretic or approximate-eigenvector argument I have not worked out for the bilateral, weighted setting. The continuity hypothesis on w in Theorem 5.2 is almost certainly stronger than necessary; a Muckenhoupt-type A_p condition ought to suffice, at the cost of replacing Proposition 4.2 with a genuine maximal-function argument, but I have not checked the details. The Toeplitz operators T_φ of Definition 8.1 were only really considered here for holomorphic symbols; a general continuous symbol φ would bring in essentially all of the machinery of Douglas [13] and Zhu [11], which felt like a separate paper rather than a short addition to this one. And it would be worth knowing how much of Theorem 3.3 survives once the annulus is replaced by a finitely connected domain with more than two boundary circles, where a single weight and a single diffeomorphism Φ_w would presumably have to be replaced by a family of local uniformizing data glued together in some way. I hope to come back to at least the first of these.

XII. ACKNOWLEDGMENT

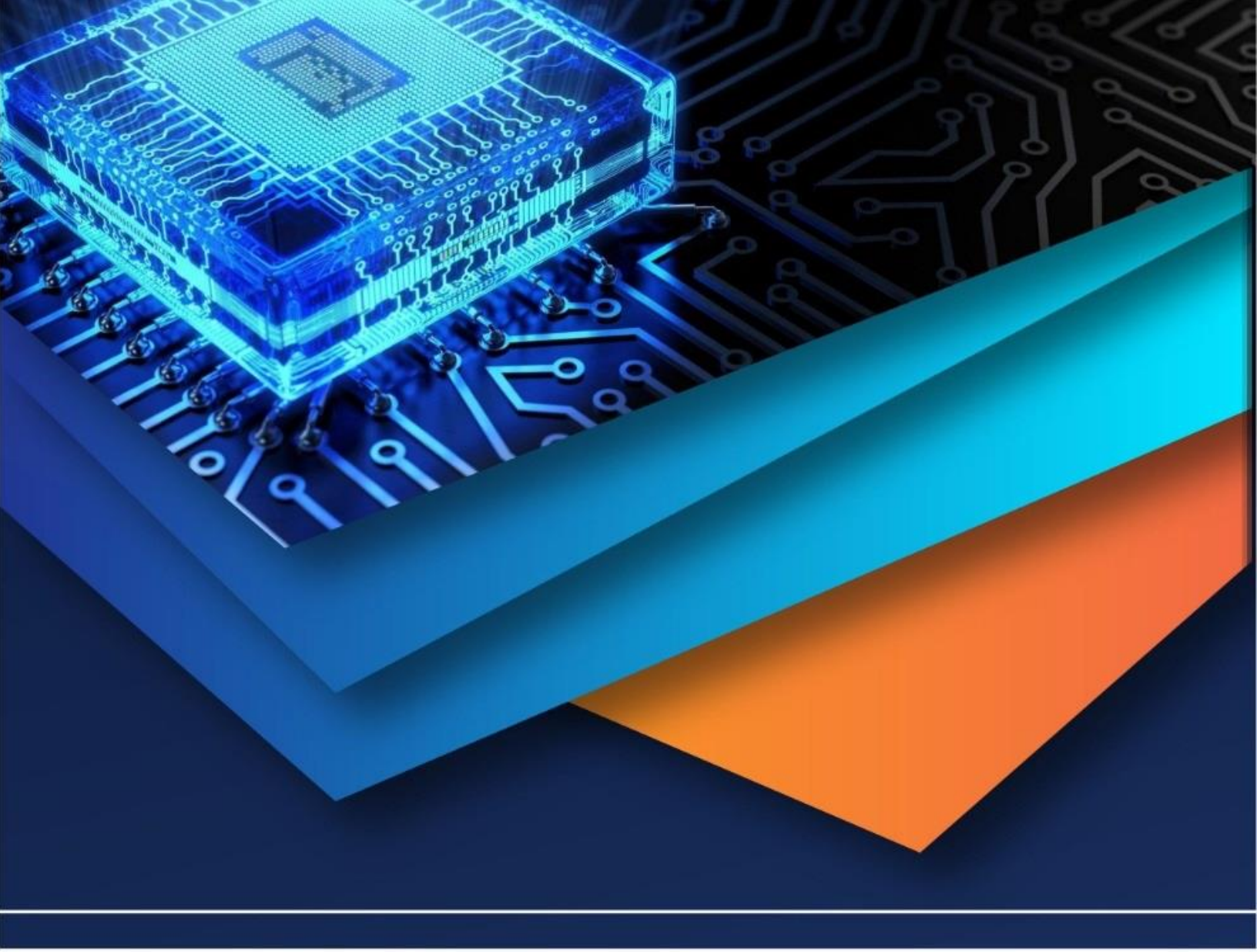
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XIII. CONFLICT OF INTEREST

The author declares that there is no conflict of interest regarding the publication of this paper.

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