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Certain Summation and Transformation Formulae for Basic Hypergeometric Series

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Abstract: In this paper, making use of certain known summation formulae an attempt has been made to establish certain very interesting summation and transformation formulae for basic hypergeometric series.
Keywords: Hypergeometric functions, Summation, Transformation, Polybasic, Converges

I. INTRODUCTION

In 1972, Verma [2] established the following very general transformation formulae:

$$\sum_{n=0}^{\infty} A_n B_n \frac{(x\omega)^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{(q,\gamma q^n;q)_n} \sum_{k=0}^{\infty} \frac{(\alpha,\beta;q)_{n+k} B_{n+k} x^k}{(q;q)_k (\gamma q^{2n+1};q)_k} \sum_{j=0}^{n} \frac{(q^{-n},\gamma q^n;q)_j A_j (\omega q)^j}{(q,\alpha,\beta;q)_j} \tag{1.1}$$

Making use of certain known summation formulae due to Verma and Jain [4] and the expansion formulae (1.1) , an attempt has been made to establish certain very interesting summation and transformation formulae for basic hypergeometric series.

II. DEFINITIONS AND NOTATIONS

The following result will be required in our analysis:-

$${}_2\Phi_1\left[\begin{matrix} a,b;q;\dfrac{c}{ab} \\ cq \end{matrix}\right] = \frac{\left(\dfrac{cq}{a},\dfrac{cq}{b};q\right)_{\infty}}{\left(cq,\dfrac{cq}{ab};q\right)_{\infty}} \left\{\frac{ab(1+c)-(a+b)c}{ab-c}\right\} \tag{2.1}$$

$${}_4\Phi_3\left[\begin{matrix} q^{-n},x^2y^2q^{1+n},x,-xq;q;q \\ xyq,-xyq,x^2q \end{matrix}\right] = \frac{x^n (q;q)_n (x^2q^2;q^2)_m (y^2q^2;q^2)_m}{(x^2q;q)_n (x^2y^2q^2;q^2)_m (q^2;q^2)_m} \tag{2.2}$$

Where m is the greatest integer $\leq n/2$

$${}_3\Phi_2\left[\begin{matrix} q^{-n},x^2q^{1+n},0;q;q \\ xy,-xy \end{matrix}\right] = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^n q^{n(n+1)} x^{2n} \dfrac{(q;q^2)_n}{(x^2q^2;q^2)_n} & \text{if } n \text{ is even} \end{cases} \tag{2.3}$$

$${}_5\Phi_4\left[\begin{matrix} x,aq^{1+n},\left(\dfrac{aq}{b}\right)^{1/2},-\left(\dfrac{aq}{b}\right)^{1/2},q^{-n};q;q \\ (aq)^{1/2},-(aq)^{1/2},\dfrac{aq}{b},xq \end{matrix}\right] = \frac{x^n (q;q)_n (aq/x;q)_n}{(aq;q)_n (xq;q)_n}$$

$$\times {}_6\Phi_5\left[\begin{matrix} a,q^2\sqrt{a},-q^2\sqrt{a},b,x,xq;q^2;\dfrac{aq}{bx^2} \\ \sqrt{a},-\sqrt{a},\dfrac{aq^2}{b},\dfrac{aq^2}{x},\dfrac{aq}{x} \end{matrix}\right] \text{ to } (n+1) \text{ terms,} \tag{2.4}$$

Where m is the greatest integer $< n/2$

$${}_4\Phi_3\left[\begin{matrix} x,-xq,bx^2q^{2+n},q^{-n};q;q \\ x^2q^2,xq\sqrt{b},-xq\sqrt{b} \end{matrix}\right] = \frac{x^n (q;q)_n (bxq^2;q)_n (bx^2q^3;q^2)_m (bq^2;q^2)_m (xq^2;q)_{2m}}{(xq;q)_n (bx^2q^2;q)_n (q^2;q^2)_m (x^2q^3;q^2)_m (bxq^2;q)_{2m}} \tag{2.5}$$

Where m is the greatest integer $< n/2$

$$ {}_5\Phi_4\left[\begin{matrix} x, aq^{1+n}, \left(\frac{x}{q}\right)^{1/2}, -\left(\frac{x}{q}\right)^{1/2}, q^{-n}; q; q \\ (aq)^{1/2}, -(aq)^{1/2}, \frac{x}{q}, xq \end{matrix}\right] = \frac{x^{n-m}\left(\frac{aq}{x};q\right)_n (q;q)_n (aq^2;q^2)_m (xq;q^2)_m}{q^m (aq;q)_n (xq;q)_n (q^2;q^2)_m \left(\frac{aq}{x};q^2\right)_m} \tag{2.6} $$

Where m is the greatest integer $< n/2$

III.MAIN RESULTS

In this section we shall establish our main results :

$$ {}_4\Phi_3\left[\begin{matrix} \alpha, \beta, x, -xq; q, \frac{x^2y^2q}{\alpha\beta} \\ xyq, -xyq, x^2q \end{matrix}\right] = \frac{\left(\frac{x^2y^2}{\alpha}q^2, \frac{x^2y^2}{\beta}q^2; q\right)_\infty}{\left(x^2y^2q^2, \frac{x^2y^2}{\alpha\beta}q^2; q\right)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha,\beta;q)_n (x^2y^2q^2;q)_{2n} (x^2q^2, y^2q^2; q^2)_m}{\left(x^2y^2q^{n+1}, \frac{x^2y^2}{\alpha}q^2, \frac{x^2y^2}{\beta}q^2, x^2q; q\right)_n (x^2y^2q^2, q^2; q^2)_m} $$

$$ \times\left(-\frac{x^3y^2}{\alpha\beta}\right)^n q^{n(n+1)/2} \left\{\frac{\alpha\beta(1+x^2y^2q^{2n+1}) - x^2y^2q^{n+1}(\alpha+\beta)}{\alpha\beta - x^2y^2q}\right\} \tag{3.1} $$

Where m is the greatest integer $\le n/2$

$$ {}_3\Phi_2\left[\begin{matrix} \alpha, \beta, 0; q; \frac{x^2q}{\alpha\beta} \\ xq, -xq \end{matrix}\right] = \frac{\left(\frac{x^2}{\alpha}q^2, \frac{x^2}{\beta}q^2; q\right)_\infty}{\left(x^2q^2, \frac{x^2}{\alpha\beta}q^2; q\right)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha,\beta;q)_n (x^2q^2;q)_{2n}}{\left(q, x^2q^{n+1}, \frac{x^2}{\alpha}q^2, \frac{x^2}{\beta}q^2; q\right)_n} \left(-\frac{x^2}{\alpha\beta}\right)^n q^{n(n+1)/2} $$

$$ \left\{\frac{\alpha\beta(1+x^2q^{2n+1}) - x^2q^{n+1}(\alpha+\beta)}{\alpha\beta - x^2q}\right\} \times \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^n q^{n(n+1)} \frac{x^{2n}(q;q^2)_n}{(x^2q^2;q^2)_n}, & \text{if } n \text{ is even} \end{cases} \tag{3.2} $$

$$ {}_5\Phi_4\left[\begin{matrix} \alpha, \beta, x\left(\frac{aq}{b}\right)^{1/2}, -\left(\frac{aq}{b}\right)^{1/2}; q; \frac{aq}{\alpha\beta} \\ (aq)^{1/2}, -(aq)^{1/2}, \frac{aq}{b}, xq \end{matrix}\right] = \frac{\left(\frac{a}{\alpha}q^2, \frac{a}{\beta}q^2; q\right)_\infty}{\left(aq^2, \frac{a}{\alpha\beta}q^2; q\right)_\infty} $$

$$ \times\sum_{n=0}^{\infty} \frac{\left(\alpha, \beta, \frac{aq}{x}; q\right)_n (aq^2;q)_{2n}}{\left(aq^{n+1}, \frac{a}{\alpha}q^2, \frac{a}{\beta}q^2, aq, xq; q\right)_n} \left(-\frac{ax}{\alpha\beta}\right)^n q^{n(n+1)/2} \left\{\frac{\alpha\beta(1+aq^{2n+1}) - aq^{n+1}(\alpha+\beta)}{\alpha\beta - aq}\right\} $$

$$ \times {}_6\Phi_5\left[\begin{matrix} a, q^2\sqrt{a}, -q^2\sqrt{a}, b, x, xq; q^2; \frac{aq}{bx^2} \\ \sqrt{a}, -\sqrt{a}, \frac{aq^2}{b}, \frac{aq^2}{x}, \frac{aq}{x} \end{matrix}\right] \text{ to } (m+1) \text{ terms,} \tag{3.3} $$

Where m is the greatest integer $\le n/2$

$$ {}_4\Phi_3\left[\begin{matrix} \alpha, \beta, x, -xq; q; \frac{bx^2q^2}{\alpha\beta} \\ x^2q^2, xq\sqrt{b}, -xq\sqrt{b} \end{matrix}\right] = \frac{\left(\frac{b}{\alpha}x^2q^3, \frac{b}{\beta}x^2q^3; q\right)_\infty}{\left(bx^2q^3, \frac{b}{\alpha\beta}x^2q^3; q\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(\alpha, \beta, bxq^2; q\right)_n \left(bx^2q^3; q\right)_{2n}}{\left(bx^2q^{n+2}, \frac{b}{\alpha}x^2q^3, \frac{b}{\beta}x^2q^3, bx^2q^2, xq; q\right)_n} $$

$$\times\frac{\left(bx^2q^3,bq^2;q^2\right)_m\left(xq^2;q\right)_{2m}}{\left(x^2q^3,q^2;q^2\right)_m(bxq^2;q)_{2m}}\left(\frac{-bx^3}{\alpha\beta}\right)q^{n(n+3)/2}\times\left\{\frac{\alpha\beta(1+bx^2q^{2n+2})-bx^2q^{n+2}(\alpha+\beta)}{\alpha\beta-bx^2q^2}\right\} \quad (3.4)$$

Where m is the greatest integer $\leq n/2$

$${}_5\Phi_4\left[\begin{matrix}\alpha,\beta,x,\sqrt{\frac{x}{q}},-\sqrt{\frac{x}{q}};q;\frac{aq}{\alpha\beta}\\ \sqrt{aq},-\sqrt{aq}\quad,\frac{x}{q},xq\end{matrix}\right]=\frac{\left(\frac{a}{\alpha}q^2,\frac{a}{\beta}q^2;q\right)_\infty}{\left(aq^2,\frac{a}{\alpha\beta}q^2;q\right)_\infty}\times$$

$$\times\sum_{n=0}^{\infty}\frac{\left(\alpha,\beta,\frac{aq}{x};q\right)_n\left(aq^2;q\right)_{2n}\left(-\frac{ax}{\alpha\beta}\right)^n(xq)^{-m}q^{n(n+1)/2}}{\left(aq^{n+1}\frac{a}{\alpha}q^2,\frac{a}{\beta}q^2,aq,xq;q\right)_n}\times$$

$$\times\frac{\left(aq^2,xq;q^2\right)_m}{\left(\frac{aq}{x},q^2,q^2\right)_m}\left\{\frac{\alpha\beta(1+aq^{2n+1})-aq^{n+1}(\alpha+\beta)}{\alpha\beta-aq}\right\} \quad (3.5)$$

Where m is the greatest integer $\leq n/2$.

$${}_6\Phi_5\left[\begin{matrix}\alpha,\beta,x,a^{1/3},ba^{1/3},b^2a^{1/3};q;\frac{aq}{\alpha\beta}\\ \sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},xq\end{matrix}\right]=\frac{\left(\frac{a}{\alpha}q^2,\frac{a}{\beta}q^2;q\right)_\infty}{\left(aq^2,\frac{a}{\alpha\beta}q^2;q\right)_\infty}\times$$

$$\times\sum_{n=0}^{\infty}\frac{\left(\alpha,\beta,\frac{aq}{x};q\right)_n\left(aq^2;q\right)_{2n}}{\left(aq^{n+1}\frac{a}{\alpha}q^2,\frac{a}{\beta}q^2,aq,xq;q\right)_n}\left(-\frac{ax}{\alpha\beta}\right)^n q^{n(n+1)/2}\times$$

$$\times\left\{\frac{\alpha\beta(1+aq^{2n+1})-aq^{n+1}(\alpha+\beta)}{\alpha\beta-aq}\right\}$$

$$\times{}_6\Phi_5\left[\begin{matrix}a,q^3\sqrt{a},-q^3\sqrt{a},x,xq,xq^2;q^3;\frac{a}{x^3}\\ \sqrt{a},-\sqrt{a},\quad\frac{aq^3}{b},\frac{aq^2}{x},\frac{aq}{x}\end{matrix}\right]\text{ to (m+1) terms,} \quad (3.6)$$

Where m is the greatest integer $\leq n/3$.

Proof of (3.1) - (3.6)

In this section we shall give the outline of the proof of (3.1) – (3.6)

1) In order to prove (3.1), let us suppose that

$$A_n=\frac{(\alpha,\beta,x,-xq;q)_n}{(xyq,-xyq,x^2q;q)_n(\omega)^n},\ B_n=\left(\frac{xy^2q}{\alpha\beta}\right)^n \text{ and } \gamma=x^2y^2q$$

In (1.1), we get;

$${}_4\Phi_3\left[\begin{matrix}\alpha,\beta,x,-xq;q;\frac{x^2y^2q}{\alpha\beta}\\ xyq,-xyq,x^2q\end{matrix}\right]={}_2\Phi_1\left[\begin{matrix}\alpha q^n,\beta q^n;q,\frac{x^2y^2q}{\alpha\beta}\\ x^2y^2q^{2n+2}\end{matrix}\right]\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(\alpha,\beta;q)_n}{(q,x^2y^2q^{n+1};q)_n}\left(-\frac{x^2y^2q}{\alpha\beta}\right)^n\times$$

$$\times {}_4\Phi_3\left[\begin{matrix} q^{-n}, x^2y^2q^{1+n}, x, -xq; q; q \\ xyq, -xyq, x^2q \end{matrix}\right]$$

Now making use of (2.1) and (2.2), we get (3.1) after some simplifications.

2) In order to prove (3.2), let us suppose that

$$A_n = \frac{(\alpha, \beta, 0; q)_n}{(xq, -xq; q)_n (\omega)^n}, B_n = \left(\frac{xq}{\alpha\beta}\right)^n \text{ and } \gamma = x^2q$$

In (1.1) , we get;

$${}_3\Phi_2\left[\begin{matrix} \alpha, \beta, 0; q; \dfrac{x^2q}{\alpha\beta} \\ xq, -xq \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(\alpha, \beta; q)_n}{(q, x^2q^{n+1}; q)_n}\left(-\frac{x^2q}{\alpha\beta}\right)^n \times {}_2\Phi_1\left[\begin{matrix} \alpha q^n, \beta q^n; q, \dfrac{x^2q}{\alpha\beta} \\ x^2q^{2n+2} \end{matrix}\right] \times {}_3\Phi_2\left[\begin{matrix} q^{-n}, x^2q^{n+1}, 0; q; q \\ xq, -xq \end{matrix}\right]$$

Now making use of (2.1) and (2.3), we get (3.2) after some simplifications.

3) In order to prove (3.3), let us suppose that

$$A_n = \frac{\left(\alpha, \beta, x, \sqrt{\frac{aq}{b}}, -\sqrt{\frac{aq}{b}}; q\right)_n}{\left(\sqrt{aq}, -\sqrt{aq}, \frac{aq}{b}, xq; q\right)_n (\omega)^n}, B_n = \left(\frac{aq}{x\alpha\beta}\right)^n \text{ and } \gamma = aq$$

In (1.1) , we get ;

$${}_5\Phi_4\left[\begin{matrix} \alpha, \beta, x, \sqrt{\dfrac{aq}{b}}, -\sqrt{\dfrac{aq}{b}}; q; \dfrac{aq}{\alpha\beta} \\ \sqrt{aq}, -\sqrt{aq}, \dfrac{aq}{b}, xq \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(\alpha, \beta; q)_n}{(q, aq^{n+1}; q)_n}\left(-\frac{aq}{\alpha\beta}\right)^n {}_2\Phi_1\left[\begin{matrix} \alpha q^n, \beta q^n; q, \dfrac{aq}{\alpha\beta} \\ aq^{2n+2} \end{matrix}\right] \times$$

$$\times {}_5\Phi_4\left[\begin{matrix} x, aq^{n+1}, \sqrt{\dfrac{aq}{b}}, -\sqrt{\dfrac{aq}{b}}, q^{-n}; q; q \\ \sqrt{aq}, -\sqrt{aq}, \dfrac{aq}{b}, xq \end{matrix}\right]$$

Now making use of (2.1) and (2.4), we get (3.3) after some simplifications.

4) In order to prove (3.4), let us suppose that

$$A_n = \frac{(\alpha, \beta, x, -xq; q)_n}{(x^2q^2, xq\sqrt{b} - xq\sqrt{b}; q)_n (\omega)^n}, B_n = \left(\frac{bxq^2}{\alpha\beta}\right)^n \text{ and } \gamma = bx^2q^2$$

In (1.1), we get;

$${}_4\Phi_3\left[\begin{matrix} \alpha, \beta, x, -xq; q; \dfrac{bx^2q^2}{\alpha\beta} \\ x^2q^2, xq\sqrt{b}, -xq\sqrt{b} \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(\alpha, \beta; q)_n}{(q, bx^2q^{n+2}; q)_n}\left(-\frac{bx^2q^2}{\alpha\beta}\right)^n {}_2\Phi_1\left[\begin{matrix} \alpha q^n, \beta q^n; q, \dfrac{bx^2q^2}{\alpha\beta} \\ bx^2q^{2n+3} \end{matrix}\right] \times$$

$$\times {}_4\Phi_3\left[\begin{matrix} x, -xq, bx^2q^{2+n}, q^{-n}; q; q \\ x^2q^2, xq\sqrt{b}, -xq\sqrt{b} \end{matrix}\right]$$

Now making use of (2.1) and (2.5), we get (3.4) after some simplifications.

5) In order to prove (3.5), let us suppose that

$$A_n = \frac{\left(\alpha, \beta, x, \sqrt{\frac{x}{q}}, -\sqrt{\frac{x}{q}}; q\right)_n}{\left(\sqrt{aq}, -\sqrt{aq}, \frac{x}{q}, xq; q\right)_n (\omega)^n}, B_n = \left(\frac{aq}{x\alpha\beta}\right)^n \text{ and } \gamma = aq$$

In (1.1), we get;

$$
{}_5\Phi_4\left[\begin{matrix}\alpha,\beta,x,\sqrt{\frac{x}{q}},-\sqrt{\frac{x}{q}};q;\frac{aq}{\alpha\beta}\\ \sqrt{aq},-\sqrt{aq},\frac{x}{q},xq\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(\alpha,\beta;q)_n}{(q,aq^{n+1};q)_n}\left(-\frac{aq}{\alpha\beta}\right)^n {}_2\Phi_1\left[\begin{matrix}\alpha q^n,\beta q^n;q,\frac{aq}{\alpha\beta}\\ aq^{2n+2}\end{matrix}\right]\times
$$

$$
\times{}_5\Phi_4\left[\begin{matrix}x,aq^{n+1},\sqrt{\frac{x}{q}},-\sqrt{\frac{x}{q}},q^{-n};q;q\\ \sqrt{aq},-\sqrt{aq},\frac{x}{q},xq\end{matrix}\right]
$$

Now making use of (2.1) and (2.6), we get (3.5) after some simplifications.

6) In order to prove (3.6), let us suppose that

$$
A_n=\frac{(\alpha,\beta,x,a^{1/3},ba^{1/3};q)_n}{(\sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},xq;q)_n(\omega)^n},\ B_n=\left(\frac{aq}{x\alpha\beta}\right)^n \text{ and } \gamma=aq
$$

In (1.1), we get ;

$$
{}_6\Phi_5\left[\begin{matrix}\alpha,\beta,x,a^{1/3},ba^{1/3},b^2a^{1/3};q;\frac{aq}{\alpha\beta}\\ \sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},xq\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(\alpha,\beta;q)_n}{(q,aq^{n+1};q)_n}\left(-\frac{aq}{\alpha\beta}\right)^n {}_2\Phi_1\left[\begin{matrix}\alpha q^n,\beta q^n;q,\frac{aq}{\alpha\beta}\\ aq^{2n+2}\end{matrix}\right]\times
$$

$$
\times{}_6\Phi_5\left[\begin{matrix}a^{1/3},ba^{1/3},b^2a^{1/3},x,aq^{n+1},q^{-n};q;q\\ \sqrt{a},-\sqrt{a},\sqrt{aq},-\sqrt{aq},xq\end{matrix}\right].
$$

Now making use of (2.1) and (2.7), we get (3.6) after some simplifications.

IV. CONCLUSION

In this paper, an attempt has been made to establish six certain transformation formulae for basic hypergeometric series by making use of the identity (1.1)

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