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# Certain Summation and Transformation Formulae for Basic Hypergeometric Series

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**Abstract:** In this paper, making use of certain known summation formulae an attempt has been made to establish certain very interesting summation and transformation formulae for basic hypergeometric series.

**Keywords:** Hypergeometric functions, Summation, Transformation, Polybasic, Converges

## I. INTRODUCTION

In 1972, Verma [2] established the following very general transformation formulae:

$$\sum_{n=0}^{\infty} A_n B_n \frac{(x\omega)^n}{(q; q)_n} = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{(q, \gamma q^n; q)_n} \sum_{k=0}^{\infty} \frac{(\alpha, \beta; q)_{n+k} B_{n+k} x^k}{(q; q)_k (\gamma q^{2n+1}; q)_k} \sum_{j=0}^n \frac{(q^{-n}, \gamma q^n; q)_j A_j (\omega q)^j}{(q, \alpha, \beta; q)_j} \quad (1.1)$$

Making use of certain known summation formulae due to Verma and Jain [4] and the expansion formulae (1.1), an attempt has been made to establish certain very interesting summation and transformation formulae for basic hypergeometric series.

## II. DEFINITIONS AND NOTATIONS

The following result will be required in our analysis:-

$${}_2\Phi_1 \left[ \begin{matrix} a, b; q; \frac{c}{ab} \\ cq \end{matrix} \right] = \frac{\left( \frac{cq}{a}, \frac{cq}{b}; q \right)_{\infty}}{\left( cq, \frac{cq}{ab}; q \right)_{\infty}} \left\{ \frac{ab(1+c) - (a+b)c}{ab-c} \right\} \quad (2.1)$$

$${}_4\Phi_3 \left[ \begin{matrix} q^{-n}, x^2 y^2 q^{1+n}, x, -xq; q; q \\ xyq, -xyq, x^2 q \end{matrix} \right] = \frac{x^n (q; q)_n (x^2 q^2; q^2)_m (y^2 q^2; q^2)_m}{(x^2 q; q)_n (x^2 y^2 q^2; q^2)_m (q^2; q^2)_m} \quad (2.2)$$

Where m is the greatest integer  $\leq n/2$

$${}_3\Phi_2 \left[ \begin{matrix} q^{-n}, x^2 q^{1+n}, 0; q; q \\ xy, -xy \end{matrix} \right] = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^n q^{n(n+1)} x^{2n} \frac{(q; q^2)_n}{(x^2 q^2; q^2)_n} & \text{if } n \text{ is even} \end{cases} \quad (2.3)$$

$${}_5\Phi_4 \left[ \begin{matrix} x, aq^{1+n}, \left( \frac{aq}{b} \right)^{1/2}, -\left( \frac{aq}{b} \right)^{1/2}, q^{-n}; q; q \\ (aq)^{1/2}, -(aq)^{1/2}, \frac{aq}{b}, xq \end{matrix} \right] = \frac{x^n (q; q)_n (aq/x; q)_n}{(aq; q)_n (xq; q)_n}$$

$${}_6\Phi_5 \left[ \begin{matrix} a, q^2 \sqrt{a}, -q^2 \sqrt{a}, b, x, xq; q^2; \frac{aq}{bx^2} \\ \sqrt{a}, -\sqrt{a}, \frac{aq^2}{b}, \frac{aq^2}{x}, \frac{aq}{x} \end{matrix} \right] \text{ to } (n+1) \text{ terms,} \quad (2.4)$$

Where m is the greatest integer  $< n/2$

$${}_4\Phi_3 \left[ \begin{matrix} x, -xq, bx^2 q^{2+n}, q^{-n}; q; q \\ x^2 q^2, xq\sqrt{b}, -xq\sqrt{b} \end{matrix} \right] = \frac{x^n (q; q)_n (bx^2 q^2; q)_n (bx^2 q^3; q^2)_m (bq^2; q^2)_m (xq^2; q)_{2m}}{(xq; q)_n (bx^2 q^2; q)_n (q^2; q^2)_m (x^2 q^3; q^2)_m (bx^2 q^2; q)_{2m}} \quad (2.5)$$

Where m is the greatest integer  $< n/2$

$${}_5\Phi_4 \left[ \begin{matrix} x, aq^{1+n}, \left(\frac{x}{q}\right)^{1/2}, -\left(\frac{x}{q}\right)^{1/2}, q^{-n}; q; q \\ (aq)^{1/2}, -(aq)^{1/2}, \frac{x}{q}, xq \end{matrix} \right] = \frac{x^{n-m} \left(\frac{aq}{x}; q\right)_n (q; q)_n (aq^2; q^2)_m (xq; q^2)_m}{q^m (aq; q)_n (xq; q)_n (q^2; q^2)_m \left(\frac{aq}{x}; q^2\right)_m} \quad (2.6)$$

Where m is the greatest integer  $< n / 2$

### III.MAIN RESULTS

In this section we shall establish our main results :

$${}_4\Phi_3 \left[ \begin{matrix} \alpha, \beta, x, -xq; q, \frac{x^2y^2q}{\alpha\beta} \\ xyq, -xyq, x^2q \end{matrix} \right] = \frac{\left(\frac{x^2y^2}{\alpha}q^2, \frac{x^2y^2}{\beta}q^2; q\right)_\infty}{\left(x^2y^2q^2, \frac{x^2y^2}{\alpha\beta}q^2; q\right)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n (x^2y^2q^2; q)_{2n} (x^2q^2, y^2q^2; q^2)_m}{\left(x^2y^2q^{n+1}, \frac{x^2y^2}{\alpha}q^2, \frac{x^2y^2}{\beta}q^2, x^2q; q\right)_n (x^2y^2q^2, q^2; q^2)_m} \\ \times \left(-\frac{x^3y^2}{\alpha\beta}\right)^n q^{n(n+1)/2} \left\{ \frac{\alpha\beta(1+x^2y^2q^{2n+1}) - x^2y^2q^{n+1}(\alpha+\beta)}{\alpha\beta - x^2y^2q} \right\} \quad (3.1)$$

Where m is the greatest integer  $\leq n / 2$

$${}_3\Phi_2 \left[ \begin{matrix} \alpha, \beta, 0; q; \frac{x^2q}{\alpha\beta} \\ xq, -xq \end{matrix} \right] = \frac{\left(\frac{x^2}{\alpha}q^2, \frac{x^2}{\beta}q^2; q\right)_\infty}{\left(x^2q^2, \frac{x^2}{\alpha\beta}q^2; q\right)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha, \beta; q)_n (x^2q^2; q)_{2n}}{\left(q, x^2q^{n+1}, \frac{x^2}{\alpha}q^2, \frac{x^2}{\beta}q^2; q\right)_n} \left(-\frac{x^2}{\alpha\beta}\right)^n q^{n(n+1)/2} \\ \left\{ \frac{\alpha\beta(1+x^2q^{2n+1}) - x^2q^{n+1}(\alpha+\beta)}{\alpha\beta - x^2q} \right\} \times \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^n q^{n(n+1)} \frac{x^{2n}(q; q^2)_n}{(x^2q^2; q^2)_n}, & \text{if } n \text{ is even} \end{cases} \quad (3.2)$$

$${}_5\Phi_4 \left[ \begin{matrix} \alpha, \beta, x \left(\frac{aq}{b}\right)^{1/2}, -\left(\frac{aq}{b}\right)^{1/2}; q; \frac{aq}{\alpha\beta} \\ (aq)^{1/2}, -(aq)^{1/2}, \frac{aq}{b}, xq \end{matrix} \right] = \frac{\left(\frac{a}{\alpha}q^2, \frac{a}{\beta}q^2; q\right)_\infty}{\left(aq^2, \frac{a}{\alpha\beta}q^2; q\right)_\infty} \\ \times \sum_{n=0}^{\infty} \frac{\left(\alpha, \beta, \frac{aq}{x}; q\right)_n (aq^2; q)_{2n}}{\left(aq^{n+1}, \frac{a}{\alpha}q^2, \frac{a}{\beta}q^2, aq, xq; q\right)_n} \left(-\frac{ax}{\alpha\beta}\right)^n q^{n(n+1)/2} \left\{ \frac{\alpha\beta(1+aq^{2n+1}) - aq^{n+1}(\alpha+\beta)}{\alpha\beta - aq} \right\} \\ \times {}_6\Phi_5 \left[ \begin{matrix} a, q^2\sqrt{a}, -q^2\sqrt{a}, b, x, xq; q^2; \frac{aq}{bx^2} \\ \sqrt{a}, -\sqrt{a}, \frac{aq^2}{b}, \frac{aq^2}{x}, \frac{aq}{x} \end{matrix} \right] \text{ to } (m+1) \text{ terms,} \quad (3.3)$$

Where m is the greatest integer  $\leq n / 2$

$${}_4\Phi_3 \left[ \begin{matrix} \alpha, \beta, x, -xq; q; \frac{bx^2q^2}{\alpha\beta} \\ x^2q^2, xq\sqrt{b}, -xq\sqrt{b} \end{matrix} \right] = \frac{\left(\frac{b}{\alpha}x^2q^3, \frac{b}{\beta}x^2q^3; q\right)_\infty}{\left(bx^2q^3, \frac{b}{\alpha\beta}x^2q^3; q\right)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha, \beta, bxq^2; q)_n (bx^2q^3; q)_{2n}}{\left(bx^2q^{n+2}, \frac{b}{\alpha}x^2q^3, \frac{b}{\beta}x^2q^3, bx^2q^2, xq; q\right)_n}$$

$$\times \frac{(bx^2q^3, bq^2; q^2)_m (xq^2; q)_{2m}}{(x^2q^3, q^2; q^2)_m (bxq^2; q)_{2m}} \left( \frac{-bx^3}{\alpha\beta} \right) q^{n(n+3)/2} \times \left\{ \frac{\alpha\beta(1+bx^2q^{2n+2}) - bx^2q^{n+2}(\alpha+\beta)}{\alpha\beta - bx^2q^2} \right\} \quad (3.4)$$

Where m is the greatest integer  $\leq n/2$

$${}_5\Phi_4 \left[ \begin{matrix} \alpha, \beta, x, \sqrt{\frac{x}{q}}, -\sqrt{\frac{x}{q}}; q; \frac{aq}{\alpha\beta} \\ \sqrt{aq}, -\sqrt{aq}, \frac{x}{q}, xq \end{matrix} \right] = \frac{\left( \frac{a}{\alpha}q^2, \frac{a}{\beta}q^2; q \right)_\infty}{\left( aq^2, \frac{a}{\alpha\beta}q^2; q \right)_\infty} \times$$

$$\times \sum_{n=0}^{\infty} \frac{\left( \alpha, \beta, \frac{aq}{x}; q \right)_n (aq^2; q)_{2n} \left( -\frac{ax}{\alpha\beta} \right)^n (xq)^{-m} q^{n(n+1)/2}}{\left( aq^{n+1} \frac{a}{\alpha}q^2, \frac{a}{\beta}q^2, aq, xq; q \right)_n} \times$$

$$\times \frac{(aq^2, xq; q^2)_m}{\left( \frac{aq}{x}, q^2, q^2 \right)_m} \left\{ \frac{\alpha\beta(1+aq^{2n+1}) - aq^{n+1}(\alpha+\beta)}{\alpha\beta - aq} \right\} \quad (3.5)$$

Where m is the greatest integer  $\leq n/2$ .

$${}_6\Phi_5 \left[ \begin{matrix} \alpha, \beta, x, a^{1/3}, ba^{1/3}, b^2a^{1/3}; q; \frac{aq}{\alpha\beta} \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, xq \end{matrix} \right] = \frac{\left( \frac{a}{\alpha}q^2, \frac{a}{\beta}q^2; q \right)_\infty}{\left( aq^2, \frac{a}{\alpha\beta}q^2; q \right)_\infty} \times$$

$$\times \sum_{n=0}^{\infty} \frac{\left( \alpha, \beta, \frac{aq}{x}; q \right)_n (aq^2; q)_{2n} \left( -\frac{ax}{\alpha\beta} \right)^n q^{n(n+1)/2}}{\left( aq^{n+1} \frac{a}{\alpha}q^2, \frac{a}{\beta}q^2, aq, xq; q \right)_n} \times$$

$$\times \left\{ \frac{\alpha\beta(1+aq^{2n+1}) - aq^{n+1}(\alpha+\beta)}{\alpha\beta - aq} \right\}$$

$$\times {}_6\Phi_5 \left[ \begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, x, xq, xq^2; q^3; \frac{a}{x^3} \\ \sqrt{a}, -\sqrt{a}, \frac{aq^3}{b}, \frac{aq^2}{x}, \frac{aq}{x} \end{matrix} \right] \text{ to } (m+1) \text{ terms,} \quad (3.6)$$

Where m is the greatest integer  $\leq n/3$ .

Proof of (3.1) - (3.6)

In this section we shall give the outline of the proof of (3.1) – (3.6)

1) In order to prove (3.1), let us suppose that

$$A_n = \frac{(\alpha, \beta, x, -xq; q)_n}{(xyq, -xyq, x^2q; q)_n} (\omega)^n, B_n = \left( \frac{xy^2q}{\alpha\beta} \right)^n \text{ and } \gamma = x^2y^2q$$

In (1.1), we get;

$${}_4\Phi_3 \left[ \begin{matrix} \alpha, \beta, x, -xq; q; \frac{x^2y^2q}{\alpha\beta} \\ xyq, -xyq, x^2q \end{matrix} \right] = {}_2\Phi_1 \left[ \begin{matrix} \alpha q^n, \beta q^n; q, \frac{x^2y^2q}{\alpha\beta} \\ x^2y^2q^{2n+2} \end{matrix} \right] \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (\alpha, \beta; q)_n}{(q, x^2y^2q^{n+1}; q)_n} \left( -\frac{x^2y^2q}{\alpha\beta} \right)^n \times$$

$$\times {}_4\Phi_3 \left[ \begin{matrix} q^{-n}, x^2 y^2 q^{1+n}, x, -xq; q; q \\ xyq, -xyq, x^2 q \end{matrix} \right]$$

Now making use of (2.1) and (2.2), we get (3.1) after some simplifications.

2) In order to prove (3.2), let us suppose that

$$A_n = \frac{(\alpha, \beta, 0; q)_n}{(xq, -xq; q)_n (\omega)^n}, B_n = \left( \frac{xq}{\alpha\beta} \right)^n \text{ and } \gamma = x^2 q$$

In (1.1), we get;

$${}_3\Phi_2 \left[ \begin{matrix} \alpha, \beta, 0; q; \frac{x^2 q}{\alpha\beta} \\ xq, -xq \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (\alpha, \beta; q)_n}{(q, x^2 q^{n+1}; q)_n} \left( -\frac{x^2 q}{\alpha\beta} \right)^n \times {}_2\Phi_1 \left[ \begin{matrix} \alpha q^n, \beta q^n; q, \frac{x^2 q}{\alpha\beta} \\ x^2 q^{2n+2} \end{matrix} \right] \times {}_3\Phi_2 \left[ \begin{matrix} q^{-n}, x^2 q^{n+1}, 0; q; q \\ xq, -xq \end{matrix} \right]$$

Now making use of (2.1) and (2.3), we get (3.2) after some simplifications.

3) In order to prove (3.3), let us suppose that

$$A_n = \frac{(\alpha, \beta, x, \sqrt{\frac{aq}{b}}; -\sqrt{\frac{aq}{b}}; q)_n}{(\sqrt{aq}, -\sqrt{aq}, \frac{aq}{b}, xq; q)_n (\omega)^n}, B_n = \left( \frac{aq}{x\alpha\beta} \right)^n \text{ and } \gamma = aq$$

In (1.1), we get ;

$${}_5\Phi_4 \left[ \begin{matrix} \alpha, \beta, x, \sqrt{\frac{aq}{b}}, -\sqrt{\frac{aq}{b}}; q; \frac{aq}{\alpha\beta} \\ \sqrt{aq}, -\sqrt{aq}, \frac{aq}{b}, xq \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (\alpha, \beta; q)_n}{(q, aq^{n+1}; q)_n} \left( -\frac{aq}{\alpha\beta} \right)^n \times {}_2\Phi_1 \left[ \begin{matrix} \alpha q^n, \beta q^n; q, \frac{aq}{\alpha\beta} \\ aq^{2n+2} \end{matrix} \right] \times$$

$$\times {}_5\Phi_4 \left[ \begin{matrix} x, aq^{n+1}, \sqrt{\frac{aq}{b}}, -\sqrt{\frac{aq}{b}}, q^{-n}; q; q \\ \sqrt{aq}, -\sqrt{aq}, \frac{aq}{b}, xq \end{matrix} \right]$$

Now making use of (2.1) and (2.4), we get (3.3) after some simplifications.

4) In order to prove (3.4), let us suppose that

$$A_n = \frac{(\alpha, \beta, x, -xq; q)_n}{(x^2 q^2, xq\sqrt{b} - xq\sqrt{b}; q)_n (\omega)^n}, B_n = \left( \frac{bxq^2}{\alpha\beta} \right)^n \text{ and } \gamma = bx^2 q^2$$

In (1.1), we get;

$${}_4\Phi_3 \left[ \begin{matrix} \alpha, \beta, x, -xq; q; \frac{bx^2 q^2}{\alpha\beta} \\ x^2 q^2, xq\sqrt{b}, -xq\sqrt{b} \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (\alpha, \beta; q)_n}{(q, bx^2 q^{n+2}; q)_n} \left( -\frac{bx^2 q^2}{\alpha\beta} \right)^n \times {}_2\Phi_1 \left[ \begin{matrix} \alpha q^n, \beta q^n; q, \frac{bx^2 q^2}{\alpha\beta} \\ bx^2 q^{2n+3} \end{matrix} \right] \times$$

$$\times {}_4\Phi_3 \left[ \begin{matrix} x, -xq, bx^2 q^{2+n}, q^{-n}; q; q \\ x^2 q^2, xq\sqrt{b}, -xq\sqrt{b} \end{matrix} \right]$$

Now making use of (2.1) and (2.5), we get (3.4) after some simplifications.

5) In order to prove (3.5), let us suppose that

$$A_n = \frac{(\alpha, \beta, x, \sqrt{\frac{x}{q}}; -\sqrt{\frac{x}{q}}; q)_n}{(\sqrt{aq}, -\sqrt{aq}, \frac{x}{q}, xq; q)_n (\omega)^n}, B_n = \left( \frac{aq}{x\alpha\beta} \right)^n \text{ and } \gamma = aq$$

In (1.1), we get;

$${}_5\Phi_4 \left[ \begin{matrix} \alpha, \beta, x, \sqrt{\frac{x}{q}}, -\sqrt{\frac{x}{q}}; q; \frac{aq}{\alpha\beta} \\ \sqrt{aq}, -\sqrt{aq}, \frac{x}{q}, xq \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (\alpha, \beta; q)_n \left(-\frac{aq}{\alpha\beta}\right)^n}{(q, aq^{n+1}; q)_n} {}_2\Phi_1 \left[ \begin{matrix} \alpha q^n, \beta q^n; q; \frac{aq}{\alpha\beta} \\ aq^{2n+2} \end{matrix} \right] \times \\
 \times {}_5\Phi_4 \left[ \begin{matrix} x, aq^{n+1}, \sqrt{\frac{x}{q}}, -\sqrt{\frac{x}{q}}, q^{-n}; q; q \\ \sqrt{aq}, -\sqrt{aq}, \frac{x}{q}, xq \end{matrix} \right]$$

Now making use of (2.1) and (2.6), we get (3.5) after some simplifications.

6) In order to prove (3.6), let us suppose that

$$A_n = \frac{(\alpha, \beta, x, a^{1/3}, ba^{1/3}; q)_n}{(\sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, xq; q)_n} (\omega)^n, B_n = \left(\frac{aq}{x\alpha\beta}\right)^n \text{ and } \gamma = aq$$

In (1.1), we get ;

$${}_6\Phi_5 \left[ \begin{matrix} \alpha, \beta, x, a^{1/3}, ba^{1/3}, b^2 a^{1/3}; q; \frac{aq}{\alpha\beta} \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, xq \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (\alpha, \beta; q)_n \left(-\frac{aq}{\alpha\beta}\right)^n}{(q, aq^{n+1}; q)_n} {}_2\Phi_1 \left[ \begin{matrix} \alpha q^n, \beta q^n; q; \frac{aq}{\alpha\beta} \\ aq^{2n+2} \end{matrix} \right] \times \\
 \times {}_6\Phi_5 \left[ \begin{matrix} a^{1/3}, ba^{1/3}, b^2 a^{1/3}, x, aq^{n+1}, q^{-n}; q; q \\ \sqrt{a}, -\sqrt{a}, \sqrt{aq}, -\sqrt{aq}, xq \end{matrix} \right].$$

Now making use of (2.1) and (2.7), we get (3.6) after some simplifications.

#### IV. CONCLUSION

In this paper, an attempt has been made to establish six certain transformation formulae for basic hypergeometric series by making use of the identity (1.1)

#### V. ACKNOWLEDGEMENT

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