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Evaluation of Certain Theta Functions Using Ramanujan's Modular Equations

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Abstract: In this Paper, an attempt has been made to establish modular identities with the help of Ramanujan's modular equations and using these identities, we have evaluated certain theta functions defined by Ramanujan.
Keywords: Modular equation, Modular identities, Theta function, Product identity multiplier, Degree

I. INTRODUCTION

For real and complex q $(|q| < 1)$, let

$$[\alpha; q]_\infty = \prod_{k=0}^{\infty}(1 - \alpha q^k),$$

where α is any complex number Also,

$[a_1, a_2, \dots \dots ., a_r; q]_\infty = [a_1; q]_\infty [a_2; q]_\infty \dots \dots [a_r; q]_\infty$.

Ramanujan defined the general theta function as:

$$f(a, b) = \sum_{n=0}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}, \qquad (1.1)$$

Which by an appeal of Jacobi's triple product identity yields:

$$f(a, b) = [ab, -a, -b; ab]_\infty. \qquad (1.2)$$

The most important special cases of (1) are:

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{[-q; q]_\infty}{[q; -q]_\infty} = [q^2; q^2]_\infty [-q; q^2]_\infty^2, \qquad (1.3)$$

$$\Psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{[q^2; q^2]_\infty}{[q; q^2]_\infty}, \qquad (1.4)$$

$$f(-q) = \sum_{n=-\infty}^{\infty} (-)^n\, q^{n(3n-1)/2} = [q; q]_\infty \qquad (1.5)$$

and $x(-q) = [q; q^2]_\infty$.

Let $z_r = z(r, x)\ {}_2F_1\ [1/r, (r-1)/r; 1; x]$ (1.6)

and $$q_r = q_r\,(x) = \exp\left[-\pi \operatorname{cosec} \pi/r \frac{{}_2F_1\,[1/r, (r-1)/r; 1-x]}{{}_2F_1\,[1/r, (r-1)/r; 1; x]}\right], \qquad (1.7)$$

where r=2,3,4,6 and $|x| < 1$.

$${}_2F_1\,[a; b; c; x] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k\, k!}\, x^k,$$

with $(a)_k = a(a+1)(a+2)\dots(a+k-1);\ (a)_0 = 1$.

II. NOTATIONS AND DEFINITIONS

Let n denote a fixed natural number and assume that

$$n\frac{{}_2F_1\,[1/r, (r-1)/r; 1; 1-\alpha]}{{}_2F_1\,[1/r, (r-1)/r; 1; \alpha]} = \frac{{}_2F_1\,[1/r, (r-1)/r; 1; 1-\beta]}{{}_2F_1\,[1/r, (r-1)/r; 1; \beta]}, \qquad (2.1)$$

where $r = 2{,}3{,}4$ and 6. Then a modular equation of degree 'n' in the theory of elliptic function of signature 'r' is a relation between α and β include by (2.1). We often say that β has degree n order α and $m(r) = z(r,\alpha)/z(r,\beta)$ is called the multiplier.
We shall use the following modular equations due to Ramanujan in our analysis.

1) If β and the multiplier m have degree 3, then

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}, \qquad (2.2)$$

$$\frac{9}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha}{\beta(1-\beta)}\right)^{1/2}, \qquad (2.3)$$

2) If β and the multiplier m have degree 5, then

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}, \qquad (2.4)$$

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}, \qquad (2.5)$$

3) If β and the multiplier m have degree 7, then

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}, \qquad (2.6)$$

$$\frac{49}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha}{\beta(1-\beta)}\right)^{1/2}, - 8\left(\frac{\alpha(1-\alpha}{\beta(1-\beta)}\right)^{1/3}, \qquad (2.7)$$

4) If β and the multiplier m have degree 9, then

$$m^{1/2} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}, \qquad (2.8)$$

$$\frac{3}{m^{1/2}} = \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha}{\beta(1-\beta)}\right)^{1/8}, \qquad (2.9)$$

5) If β and the multiplier m have degree 13, then

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}, \qquad (2.10)$$

$$\frac{13}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha}{\beta(1-\beta)}\right)^{1/4} - 4\left(\frac{\alpha(1-\alpha}{\beta(1-\beta)}\right)^{1/6}, \qquad (2.11)$$

6) If β and the multiplier m have degree 25, then

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}, \qquad (2.12)$$

$$\frac{5}{\sqrt{m}} = \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha}{\beta(1-\beta)}\right)^{1/8}, - 2\left(\frac{\alpha(1-\alpha}{\beta(1-\beta)}\right)^{1/12}, \qquad (2.13)$$

We shall also make use of the following results due to Ramanujan.

$$\phi(q^4) = \frac{\sqrt{z}}{2}\left(1 + \sqrt[4]{1-x}\right) \qquad (2.14)$$

$$\psi(q^8) = \frac{\sqrt{z}}{4q}\left(1 + \sqrt[4]{1-x}\right) \qquad (2.15)$$

$$\phi\left(q^{1/2}\right) = \sqrt{z}\sqrt{1+\sqrt{x}} \qquad (2.16)$$

$$\phi\left(-q^{1/2}\right) = \sqrt{z}\sqrt{1-\sqrt{x}} \qquad (2.17)$$

$$\phi\left(q^{1/4}\right) = \sqrt{z}\left(1 + \sqrt[4]{x}\right) \qquad (2.18)$$

$$\phi\left(-q^{1/4}\right) = \sqrt{z}\left(1 - \sqrt[4]{x}\right) \qquad (2.19)$$

$$\phi(q) = \sqrt{z} \qquad (2.20)$$

From (2.14) and (2.15), we have

$$\phi(q^4) - 2q\Psi(q^8) = \sqrt{z}\ \sqrt[4]{1-x} \qquad (2.21)$$

From (2.18) and (2.19), we get

$$\phi\left(q^{1/4}\right) - \phi\left(-q^{1/4}\right) = 2\sqrt{z}\ \sqrt[4]{x} \qquad (2.22)$$

III.MAIN RESULT

Modular Identities of Degree 3.

In modular equations (2.2) and (2.3), β and the multiplier m have degree 3,so from (2.20),(2.21)and (2.22), let

$$P = \frac{\phi(q)}{\phi(q^3)} = \sqrt{m}, \qquad (3.1)$$

$$Q = \frac{\phi(q^4) - 2q\Psi(q^8)}{\phi(q^{12}) - 2q^3\Psi(q^{24})} = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/4}, \qquad (3.2)$$

$$R = \frac{\phi\left(q^{1/4}\right) - \phi\left(-q^{1/4}\right)}{\phi\left(q^{3/4}\right) - \phi\left(-q^{3/4}\right)} = \sqrt{m}\left(\frac{\alpha}{\beta}\right)^{1/4}. \qquad (3.3)$$

Also

$$S = QR = m\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4}. \qquad (3.4)$$

1) Now,

$$\frac{Q}{P} = \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} \qquad (3.5)$$

Eliminating α, β and m from (2.2) and (2.3) using (3.1) and (3.5), we get the modular identity

$$P^4Q^4 - 8P^2Q^2 + 9 = P^4 + Q^4. \qquad (3.6)$$

2) From (3.2) and (3.3) we get:

$$\frac{R}{P} = \left(\frac{\alpha}{\beta}\right)^{1/4} \qquad (3.7)$$

Eliminating α, β and m from (2.2) and (2.3) using (3.1) and (3.7), we get:

$$P^4R^4 - 8P^2R^2 + 9 = P^4 + R^4. \qquad (3.8)$$

3) $\frac{S}{P^2} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4}$, (3.9)

So, eliminating α, β and m from (2.2) and (2.3) using (3.1) and (3.9), we get:

$$P^4S^2 + P^4 = S^2 + 9$$

Modular Identities of Degree 5

In modular equations (2.4) and (2.5), β and the multiplier m have degree 5, so from (2.20), (2.21) and (2.22), let us take:

$$P = \frac{\phi(q)}{\phi(q^5)} = \sqrt{m}, \tag{3.10}$$

$$Q = \frac{\phi(q^4) - 2q\Psi(q^8)}{\phi(q^{20}) - 2q^5\Psi(q^{40})} = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/4} \tag{3.11}$$

$$R = \frac{\phi(q^{1/4}) - \phi(-q^{1/4})}{\phi(q^{5/4}) - \phi(-q^{5/4})}\sqrt{m}\left(\frac{\alpha}{\beta}\right)^{1/4} \tag{3.12}$$

$$S = RQ = m\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4} \tag{3.13}$$

4) From (3.10) and (3.11), we have :

$$\frac{Q}{P} = \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} \tag{3.14}$$

Eliminating α, β and m from (2.4) and (2.5) using (3.10) and (3.14), we get:

$$P^2Q^2 - 4PQ + 5 = P^2 + Q^2. \tag{3.15}$$

5) From (3.10) and (3.12), we have :

$$\frac{R}{P} = \left(\frac{\alpha}{\beta}\right)^{1/4}. \tag{3.16}$$

Eliminating α, β and m from (2.4) and (2.5) using (3.10) and (3.16), we get:

$$P^2R^2 - 4PR + 5 = P^2 + R^2. \tag{3.17}$$

6) From (3.10) and (3.13), we have

$$\frac{S}{p^2} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4} \tag{3.18}$$

Eliminating α, β and m from (2.4) and (2.5) by making use of (3.10) and (3.18), we get:

$$P^2S^2 + P^2 = S + 5 \tag{3.19}$$

Modular Identities of Degree 7.

In modular equations (2.6) and (2.7), β and the multiplier m have degree 7, so from (2.20), (2.21) and (2.22), we have:

$$P = \frac{\phi(q)}{\phi(q^7)} = \sqrt{m}, \tag{3.20}$$

$$Q = \frac{\phi(q^4) - 2q\Psi(q^8)}{\phi(q^{28}) - 2q^7\Psi(q^{56})} = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/4}, \tag{3.21}$$

$$R = \frac{\phi(q^{1/4}) - \phi(-q^{1/4})}{\phi(q^{7/4}) - \phi(-q^{7/4})} = \sqrt{m}\left(\frac{\alpha}{\beta}\right)^{1/4}. \tag{3.22}$$

$$S = QR = m\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4}. \tag{3.23}$$

7) From (3.20) and (3.23), we have:

$$\frac{S}{P^2} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4} \qquad (3.24)$$

Now, Eliminating α, β and m from (2.6) and (2.7) using (3.20) and (3.24), we get:

$$P^4S^2 + P^4 + 8P^{3/8}\,S^{2/3} = 49 + S^2 + 8P^{4/3}\,S^{4/3} \qquad (3.25)$$

Modular Identities of Degree 9.

In modular equations (2.6) and (2.7), β and the multiplier m have degree 9, so from (2.20), (2.21) and (2.22), we have:

$$P = \frac{\phi(q)}{\phi(q^9)} = \sqrt{m}, \qquad (3.26)$$

$$Q = \frac{\phi(q^4) - 2q\Psi(q^8)}{\phi(q^{36}) - 2q^9\Psi(q^{72})} = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/4} \qquad (3.27)$$

$$R = \frac{\phi(q^{1/4}) - \phi(-q^{1/4})}{\phi(q^{9/4}) - \phi(-q^{9/4})} = \sqrt{m}\left(\frac{\alpha}{\beta}\right)^{1/4}. \qquad (3.28)$$

and

$$S = QR = m\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4}. \qquad (3.29)$$

8) From (56) and (57), we have :

$$\frac{Q}{P} = \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} \qquad (3.30)$$

Eliminating α, β and m from (2.8) and (2.9) using (3.26) and (3.30), we get:

$$P\,Q - 2\sqrt{PQ} + 3 = P + Q \qquad (3.31)$$

9) From (3.26) and (3.28), we have :

$$\frac{R}{P} = \left(\frac{\alpha}{\beta}\right)^{1/4}. \qquad (3.32)$$

Eliminating α, β and m from (2.8) and (2.9) using (3.26) and (3.32), we get:

$$P\,R - 2\sqrt{PR} + 3 = P + R \qquad (3.33)$$

10) From (3.26) and (3.29), we have :

$$\frac{S}{p^2} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4} \qquad (3.34)$$

Now, Eliminating α, β and m from (2.8) and (2.9) using (3.26) and (3.34), we get:

$$P + P\sqrt{S} = 3 + \sqrt{S}. \qquad (3.35)$$

Modular Identities of Degree 13.

In modular equations (2.10) and (2.11), β and the multiplier m have degree, so from (2.20), (2.21) and (2.22), we have:

$$P = \frac{\phi(q)}{\phi(q^{13})} = \sqrt{m}, \qquad (3.36)$$

$$Q = \frac{\phi(q^4) - 2q\Psi(q^8)}{\phi(q^{52}) - 2q^{13}\Psi(q^{104})} = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/4} \quad (3.37)$$

$$R = \frac{\phi(q^{1/4}) - \phi(-q^{1/4})}{\phi(q^{9/4}) - \phi(-q^{9/4})} = \sqrt{m}\left(\frac{\alpha}{\beta}\right)^{1/4}. \quad (3.38)$$

$$S = QR = m\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4}. \quad (3.39)$$

From (3.36) and (.39), we have:

$$\frac{S}{p^2} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4} \quad (3.40)$$

Eliminating α, β and m from (2.10) and (2.11) by making use of (3.36) and (3.40), we have:

$$P^2S + P^2 + 4P^{4/3}S^{2/3} = S + 4S^{2/3}P^{2/3} + 13. \quad (3.41)$$

Modular Identities of Degree 25

In modular equations (2.12) and (2.13), β and the multiplier m have degree 25, so from (2.20), (2.21) and (2.22), we have:

$$P = \frac{\phi(q)}{\phi(q^{25})} = \sqrt{m}, \quad (3.42)$$

$$Q = \frac{\phi(q^4) - 2q\Psi(q^8)}{\phi(q^{100}) - 2q^{25}\Psi(q^{200})} = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/4} \quad (3.43)$$

$$R = \frac{\phi(q^{1/4}) - \phi(-q^{1/4})}{\phi(q^{25/4}) - \phi(-q^{25/4})} = \sqrt{m}\left(\frac{\alpha}{\beta}\right)^{1/4}. \quad (3.44)$$

And

$$S = QR = m\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4}. \quad (3.45)$$

From (3.42) and (3.45), we have

$$\frac{S}{p^2} = \left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/4} \quad (3.46)$$

Now, Eliminating α, β and m from (2.12) and (2.13) using (3.42) and (3.46), we get:

$$P\sqrt{S} + P + 2P^{2/3}S^{1/6} = \sqrt{S} + 2S^{1/3}P^{1/3} + 5 \quad (3.47)$$

IV. CONCLUSIONS

These modular identities of different degrees of theta functions are very useful in various branches of science and engineering. It is especially useful for direct calculation of problems involving quantum mechanics.

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