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Solutions of Four Point Boundary Value Problems for Non-Linear Second-Order Differential Equations

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Abstract: In this paper, we are concerned with the existence of symmetric positive solutions for second-order differential equations. Under the suitable conditions, the existence and symmetric positive solutions are established by using Krasnoselskii's fixed-point theorems.
Keywords: Boundary value problem, Symmetric positive solution, Cones, Concave, Operator

I. INTRODUCTION

There are many results about the existence and multiplicity of positive solutions for nonlinear second-order differential equations. The existence of symmetric positive solutions of second-order four –point differential equations as follows,

$$\begin{cases} -u''(t) = f(t,v), \\ -v''(t) = g(t,u), 0 \le t \le 1 \end{cases} \qquad (1.1)$$

Subject to the boundary conditions

$$\begin{cases} u(t) = u(1-t), u'(0) - u'(1) = u(\xi_1) + u(\xi_2) \\ v(t) = v(1-t), v'(0) - v'(1) = v(\xi_1) + v(\xi_2), 0 < \xi_1 < \xi_2 < 1, \end{cases} \qquad (1.2)$$

Where $f, g: [0,1] \times R^+ \to R^+$ are continuous, both $f(.,u)$ and $g(.,u)$ are symmetric on $[0,1], f(x,0) \equiv g(x,0) \equiv 0$. The arguments for establishing the symmetric positive solution of (1.1) and (1.2) involve the properties of the functions in Lemma1 that plays a key role in defining some cones. A fixed point theorem due to Krasnoselskii is applied to yield the existence of symmetric positive solution of (1.1) and (1.2).

II. NOTATIONS AND DEFINITIONS

In this section, we present some necessary definitions and preliminary lemmas that will be used in the proof of the results.

Definition 1. Let E be a real Bananch space. A nonempty closed set $P \subset E$ is called a cone of E if it satisfies the following conditions:

1) $x \in P, \lambda > 0$ implies $\lambda x \in P$;

2) $x \in P, -x \in P$ implies $x = 0$.

Definition 2. Function u is called to be concave on $[0,1]$ if $u(rt_1 + (1-r)t_2) \ge ru(t_1) + (1-r)u(t_2), r, t_1, t_2 \in [0,1]$

Definition 3. The function u is symmetric on $[0,1]$ if $u(t) = u(1-t), t \in [0,1]$.

Definition 4. The function (u,v) is called a symmetric positive solution if the equation (1.1) if u and v are symmetric and positive on $[0,1]$, and satisfy the equation (1.2).

We shall consider the real Banach space $C[0,1]$, equipped with norm $\|u\| = \max_{0 \le t \le 1}|u(t)|$.

Denote $C^+[0,1] = \{u \in C[0,1]: u(t) \ge 0, t \in [0,1]\}$.

III.MAIN RESULTS

Lemma 1. Let $y \in C[0,1]$ be symmetrical on $[0,1]$ then the four point BVP

$$\begin{cases} u''(t) + y(t) = 0, 0 < t < 1 \\ u(t) = u(1-t), u'(0) - u'(1) = u(\xi_1) + u(\xi_2) \end{cases} \qquad (3.1)$$

has a unique symmetric solution $u(t) = \int_0^1 G(t,s)y(s)ds$, where $G(t,s) = G_1(t,s) + G_2(s)$, here

$$G_1(t,s) = \begin{cases} t(1-s), 0 \le t \le s \le 1, \\ s(1-t), 0 \le s \le t \le 1, \end{cases}$$

$$G_2(s)=\begin{cases}\frac{1}{2}[(\xi_1-s)+(\xi_2-s)-\xi_1(1-s)-\xi_2(1-s)+1], 0\le s\le \xi_1\\ \frac{1}{2}[(\xi_2-s)-\xi_1(1-s)-\xi_2(1-s)+1], \xi_1\le s\le \xi_2\\ \frac{1}{2}[-\xi_1(1-s)-\xi_2(1-s)+1], \xi_2\le s\le 1.\end{cases}$$

Proof. From (3.1), we have $u''(t)=-y(t)$. For $t\in[0,1]$, integrating from 0 to t we get

$$u'(t)=-\int_0^t y(s)ds+A_1 \tag{3.2}$$

Since $u'(t)=-u'(1-t)$, we obtain that $-\int_0^t y(s)ds+A_1=\int_0^{1-t}y(s)ds-A_1$, which leads to

$$\begin{aligned}A_1&=\frac{1}{2}\int_0^t y(s)ds+\frac{1}{2}\int_0^{1-t}y(s)ds\\ &=\frac{1}{2}\int_0^t y(s)ds-\frac{1}{2}\int_0^{1-t}y(1-s)d(1-s)\\ &=\int_0^1(1-s)y(s)ds.\end{aligned}$$

Integrating again we obtain

$$u(t)=-\int_0^t(t-s)y(s)ds+t\int_0^1(1-s)y(s)ds+A_2$$

From (3.1) and (3.2) we have

$$\begin{aligned}\int_0^1 y(s)ds=&-\int_0^{\xi_1}(\xi_1-s)y(s)ds+\xi_1\int_0^1(1-s)y(s)ds+A_2\\ &-\int_0^{\xi_2}(\xi_2-s)y(s)ds+\xi_2\int_0^1(1-s)y(s)ds+A_2.\end{aligned}$$

Thus

$$\begin{aligned}A_2=&\frac{1}{2}\int_0^{\xi_1}[(\xi_1-s)+(\xi_2-s)-\xi_1(1-s)-\xi_2(1-s)+1]y(s)ds\\ &+\frac{1}{2}\int_{\xi_1}^{\xi_2}[(\xi_2-s)-\xi_1(1-s)-\xi_2(1-s)+1]y(s)ds\\ &+\frac{1}{2}\int_{\xi_2}^{1}[-\xi_1(1-s)-\xi_2(1-s)+1]\,y(s)ds.\end{aligned}$$

From the above we can obtain the BVP (3.1) has a unique symmetric solution

$$\begin{aligned}u(t)=&-\int_0^t(t-s)\,y(s)ds+t\int_0^1(1-s)\,y(s)ds\\ &+\frac{1}{2}\int_0^{\xi_1}[(\xi_1-s)+(\xi_2-s)-\xi_1(1-s)-\xi_2(1-s)+1]\,y(s)ds\\ &+\frac{1}{2}\int_{\xi_1}^{\xi_2}[(\xi_2-s)-\xi_1(1-s)-\xi_2(1-s)+1]y(s)ds\\ &+\frac{1}{2}\int_{\xi_2}^{1}[-\xi_1(1-s)-\xi_2(1-s)+1]\,y(s)ds.\\ =&\int_0^1 G_1(t,s)y(s)ds+\int_0^1 G_2(s)y(s)ds=\int_0^1 G(t,s)y(s)ds.\end{aligned}$$

This completes the proof.

Lemma 2. Let $m_{G_2}=\min[G_2(\xi_1),G_2(\xi_2)]$, $L=\frac{4m_{G2}}{4m_{G2}+1}$, then the function $G(t,s)$ satisfies $LG(s,s)\le G(t,s)$ for $t,s\in[0,1]$.

Proof. For any $t\in[0,1]$ and $s\in[0,1]$, we have

$$G(t,s) = G_1(t,s) + G_2(s) \geq G_2(s) = \frac{1}{4m_{G_2}+1}G_2(s) + \frac{4m_{G_2}}{4m_{G_2}+1}G_2(s)$$
$$\geq \frac{1}{4}.\frac{4m_{G_2}}{4m_{G_2}+1} + \frac{4m_{G_2}}{4m_{G_2}+1}G_2(s) \geq s(1-s)\frac{4m_{G_2}}{4m_{G_2}+1} + \frac{4m_{G_2}}{4m_{G_2}+1}G_2(s)$$
$$\geq LG_1(s,s) + LG_2(s) = LG(s,s).$$

It is obvious that $G(s,s) \geq G(t,s)$ for $t,s \in [0,1]$. The proof is complete.

Lemma 3. Let $y \in C^+[0,1]$, then the unique symmetric solution $u(t)$ of the BVP (3.1) is nonnegative on $[0,1]$

Proof. Let $y \in C^+[0,1]$ From the fact $u''(t) = -y(t) \leq 0, t \in [0,1]$, we have known that the graph of $u(t)$ is concave on $[0,1]$. From (3.1). We have that

$$u(0) = u(1) = \frac{1}{2}\int_0^{\xi_1}[(\xi_1 - s) + (\xi_2 - s) - \xi_1(1-s) - \xi_2(1-s) + 1]y(s)ds$$
$$+\frac{1}{2}\int_{\xi_1}^{\xi_2}[(\xi_2 - s) - \xi_1(1-s) - \xi_2(1-s) + 1]y(s)ds$$
$$+\frac{1}{2}\int_{\xi_2}^{1}[-\xi_1(1-s) - \xi_2(1-s) + 1]\,y(s)ds \geq 0.$$

Note that $(u)t$ is concave, thus $u(t) \geq 0$ for $t \in [0,1]$. This complete the proof.

Lemma 4. Let $y \in C^+[0,1]$, then the unique symmetric solution $u(t)$ of BVP (3.1) satisfies.

$$\min_{t\in[0,1]} u(t) \geq L\|u\|. \qquad (3.3)$$

Proof. For any $t \in [0,1]$, on one hand, from Lemma 2. We have that $u(t) = \int_0^1 G(t,s)y(s)ds \leq \int_0^1 G(s,s)y(s)ds$. Therefore,

$$\|u\| \leq \int_0^1 G(s,s)y(s)ds. \qquad (3.4)$$

On the other hand, for any $t \in [0,1]$, from Lemma 2. We can obtain that

$$u(t) = \int_0^1 G(t,s)y(s)ds \geq L\int_0^1 G(s,s)y(s)ds \geq L\|u\| \qquad (3.5)$$

From (3.4) and 3.5) we know that (3.3) holds. Obviously, $(u,v) \in C^2[0,1] \times C^2[0,1]$ is the solution of (1.1) and 1.(2) if and only if $(u,v) \in C\ [0,1] \times C[0,1]$ is the solution of integral equations

$$\begin{cases} (u)t = \int_0^1 G(t,s)f\big(s,v(s)\big)ds \\ (v)t = \int_0^1 G(t,s)f\big(s,u(s)\big)ds \end{cases} \qquad (3.6)$$

Integral equation (3.6) can be transferred to the non linear integral equation

$$u(t) = \int_0^1 G(t,s)f(s,\int_0^1 G(s,\xi)g\big(\xi,u(\xi)\big)d\xi)ds \qquad (3.7)$$

Let $P = \{u \in C^+[0,1]: u(t)$ is symmetric, concave on [0,1] and $\min_{0\leq t\leq 1} u(t) \geq L\,\|u\|\}$. It is obvious that P is a positive cone in $C\,[0,1]$. Define an integral operator $A: P \to C$ by.

$$Au(t) = \int_0^1 G(t,s)f(s,\int_0^1 G(s,\xi)g\big(\xi,u(\xi)\big)d\xi)ds \qquad (3.8)$$

It is easy to see that the BVP (1.1) and (1.2) has a solution $u = u(t)$ if and only if u is a fixed point of the operator A defined by (3.8).

Lemma 5. If the operator A is defined as (3.8), then $A: P \to P$ is completely continuous

Proof. It is obvious that Au is symmetric on $[0,1]$. Note that $(Au)''(t) - f\big(t,v(t)\big) \leq 0$, we have that Au is concave, and from Lemma 3, it is easily known that $Au \in C^+[0,1]$. Thus from Lemma 2 and non-negativity of f and g .

$$Au(t) = \int_0^1 G(t,s)f(s,\int_0^1 G(s,\xi)g\big(\xi,u(\xi)\big)d\xi)ds$$
$$\leq \int_0^1 G(s,s)f(s,\int_0^1 G(s,\xi)g\big(\xi,u(\xi)\big)d\xi)ds,$$

Then

$$\|Au\| \leq \int_0^1 G(s,s) f(s, \int_0^1 G(s,\xi) g(\xi, u(\xi)) d\xi) ds,$$

For another hand,

$$Au \geq L \int_0^1 G(s,s) f(s, \int_0^1 G(s,\xi) g(\xi, u(\xi)) d\xi) ds \geq L \ \|Au\|$$

Thus, $A(P) \subset P$. Since $G(t,s), f(t,u)$ and $g(t,u)$ are continuous, it is easy to know that $A: P \to P$ is completely continuous. The proof is complete.

IV. CONCLUSIONS

From this paper we conclude that under the suitable conditions, the existence and symmetric positive solutions are established and five Lemma's are proved.

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REFERENCES

[1] J. Henderson, H.B. Thompson, Multiple symmetric positive solutions for a second order of boundary value problem, Proc.Amer.Math.Soc. 128 (2000) 2373-2379.
[2] L.H. Erbe, H.Y. Wang. On the existence of positive solution of ordinary differential equation. Proc.Amer.Math.Soc. 120(1994) 743-748.
[3] R.Y. Ma. Multiple nonnegative solutions of second –order system of boundary value problem. Non linear Anal. 42 (2002) 1003-1010.
[4] L.H. Erbe, H.Y. Wang. On the existence of positive solutions of ordinary differential equations. Proc. Amer.Math.Soc.120 (1994) 743-748.
[5] L. Hu, L.L Wang. Multiple positive solutions of boundary value problems for system of linear second-order differential equations. J. Math. Anal Appl. 335(2007) 1052-1060.
[6] G. Infantea, M. Zima: Positive solutions of multi-point boundary value problems at resonance, Nonlinear Anal. 69, 2458-2171 (2008).
[7] D.J. Guo, V. Lakshmikantham: Non Linear Problems in Abstract Cones. Academic Press, New York (1988).