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Legendre Polynomials and Their Applications in Study of One-Dimensional Transport Equation and Scattering Kernel

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Abstract: Ordinary differential equation is frequently found in physics and other technical fields. In particular it occurs when solving Laplace's equation (and related partial differential equations) in spherical co-ordinates. The Legendre's equation and Legendre's polynomials are used broadly where the directional dependence of some quantity is treated openly such as particular transport problems. In this paper, we describe a new scattering kernel and general theoretical scheme for the evaluation of the discrete and continuum eigenvalue spectrum in one dimensional slab geometry neutron transport equation. Firstly, some useful properties of Legendre polynomial which revealed during the definition of new scattering kernel are discussed. By using the scattering kernel in one-dimensional neutron transport equation we obtained an integral equation for angular part of the angular flux.

Keywords: Legendre polynomials, Transport equation, scattering kernel, Normalization, Eigenvalue

I. INTRODUCTION

This In applied science almost all researcher encounter some special classical orthogonal functions such as Legendre, Hermite and Laguerre polynomials. Along these, Legendre polynomials have an extensive usage area, particularly in physics and engineering. For example, Legendre and Associate Legendre polynomials are widely used in determination of wave functions of electrons in the orbits of an atom and in the determination of potential functions in the spherical symmetric geometry etc. Also nuclear reactor physics, Legendre polynomials have an extraordinary importance. Analytical and numerical computations neutron fluxes in a given domain are done by using two main methods called P_N and S_N methods.

II. NOTATIONS AND DEFINITIONS

Let us define two unit vectors in three- dimensional Cartesian geometry as

$$\hat{\Omega} = \sin(\theta)\cos(\varphi)\,\hat{\imath} + \sin(\theta)\sin(\varphi)\,\hat{\jmath} + \cos(\theta)\,\hat{k}, \qquad (2.1)$$

$$\hat{\Omega}' = \sin(\theta')\cos(\varphi')\,\hat{\imath} + \sin(\theta')\sin(\varphi')\,\hat{\jmath} + \cos(\theta')\,\hat{k}. \qquad (2.2)$$

By using (2.1) and (2.2) and choosing $x = \cos(\theta)$, $y = \cos(\theta')$, one can deduce that

$$\mu_0 = \cos(\theta_0) = \hat{\Omega}\,.\hat{\Omega'} = \mathrm{xy} + \sqrt{(1-x^2)(1-y^2)}\cos(\varphi-\varphi') \qquad (2.3)$$

By definition of x, y it is clear that $-1 \le x, y \le 1$ and then we have $-1 \le \mu_0 \le 1$. Generating function and the addition theorem of Legendre polynomials of the first kind are given by respectively.

$$\frac{1}{\sqrt{1-2\mu_0 t + t^2}} = \sum_{n=0}^{\infty} t^n P_n(\mu_0), |t| \le 1, \qquad (2.4)$$

$$P_n(\mu_0) = P_n(x)P_n(y) + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(x)P_n^m(y)\cos m\ (\varphi-\varphi') \qquad (2.5)$$

Multiplying both sides of (2.4) by $d\varphi'$ and using the additional theorem (2.4) then integrating in the interval $[0,2\pi]$, we can get

$$\sum_{n=0}^{\infty} t^n P_n(x)P_n(y) = \frac{2K(k)}{\pi\sqrt{a-b}}, k = \sqrt{\frac{2b}{b-a}},\ \ a = 1-2xyt+t^2,\ \ b = -2t\sqrt{1-x^2}\sqrt{1-y^2} \qquad (2.6)$$

$$-1 \le x, y, t \le 1$$

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Where $2K(k)/\pi\sqrt{a-b}$ is the generating function of the product $P_n(x)\,P_n(y)$ as seen in the above equation and $K(k)$ is the complete elliptic integral of the first kind that will be defined later.
By letting $y = 1$ in (2.6) and using the fact that $P_n(1) = 1$ and $K(0) = \pi/2$, we arrive at well known result that is, generating the function of Legendre polynomials of the first kind

$$\sum_{n=0}^{\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt+t^2}}, |t| \le 1. \qquad (2.7)$$

We can deduce from here an interesting application of this equation by setting $y = x$ in (2.6) and result is

$$\sum_{n=0}^{\infty} t^n P_n^2(x) = \frac{2K\left(2\sqrt{\frac{t(1-x^2)}{1-2(2x^2-1)t+t^2}}\right)}{\pi\sqrt{1-2(2x^2-1)t+t^2}} \qquad (2.8)$$

And one can utilize this expression as a generating function of $P_n^2(x)$ and furthermore integrating with respect to x in the interval $[-1,1]$ we find the result

$$\int_{-1}^{1} \frac{uK(2u)}{\sqrt{t(1-x^2)}} dx = \pi \sum_{n=0}^{\infty} \frac{t^n}{2n+1} = \frac{\pi}{2\sqrt{t}} \ln\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right), -1 \le t \le 1, \qquad (2.9)$$

Where $u = \sqrt{t(1-x^2)/(1-2(2x^2-1)t+t^2)}$
If we expand the function $1/(v-\mu_0)$ in terms of Legendre polynomials of the first kind, we find the result

$$\frac{1}{v-\mu_0} = \sum_{n=0}^{\infty} (2n+1) P_n(\mu_0) Q_n(v), \quad |\mu_0| \le 1, \ |v| > 1 \qquad (2.10)$$

By using (2.3) and (2.5) in (2.10) and integrating with respect to φ' in the interval $[0,2\pi]$, we find that

$$\frac{1}{\sqrt{v^2+x^2+y^2-2vxy-1}} = \sum_{n=0}^{\infty} (2n+1) P_n(x) P_n(y) Q_n(v), -1 \le x, y \le 1, \ |v| > 1 \quad (2.11)$$

Multiplying both sides of (2.11) by $1/\sqrt{1-2yt+t^2}$ and integrating with respect to y in the interval $[-1,1]$ we find that

$$\sum_{n=0}^{\infty} t^n P_n(x) Q_n(v) = \frac{2}{\sqrt{z_1}}\left(F\left(\sqrt{\frac{z_2}{z_3}}, 2\sqrt{\frac{z_4}{z_1}}\right) - F\left(\sqrt{\frac{z_5}{z_3}}, 2\sqrt{\frac{z_4}{z_1}}\right)\right), -1 \le x, t \le 1, |v| > 1, \qquad (2.12)$$

Where $F(z,k)$ is the first incomplete elliptic integral that will be defined later and

$$z_1 = -1 + 2(xv - \sqrt{x^2-1}\sqrt{v^2-1}) - t^2, z_2 = 1 - xv + \sqrt{x^2-1}\sqrt{v^2-1},$$
$$z_3 = 2\sqrt{x^2-1}\sqrt{v^2-1}, \ z_4 = -t\sqrt{x^2-1}\sqrt{v^2-1}, z_5 = -1 - xv + \sqrt{x^2-1}\sqrt{v^2-1}$$

Furthermore if we set $x = 1$ in (2.10) and then multiplying both sides by $1/\sqrt{1-2yt+t^2}$ and integrating with respect to y in the interval $[-1,1]$ we find the generating function of Legendre polynomials of the second kind $Q_n(v)$ as

$$\sum_{n=0}^{\infty} t^n Q_n(v) = \frac{1}{2\sqrt{1-2vt+t^2}} \, In\left(\frac{v-t+\sqrt{1-2vt+t^2}}{v-t-\sqrt{1-2vt+t^2}}\right), |t| \le 1, \ |v|.1 \qquad (2.13)$$

It is also possible to find some other properties of Legendre polynomials of the second kind using (2.11). For instance, letting $y = 1$ in (2.11) and then multiplying both sides by $1/2 \ (v'-x)$ and integrating the result over x in the interval $[-1,1]$ we have

$$\sum_{n=0}^{\infty} (2n+1) Q_n(v') Q_n(v) = \frac{1}{2(v-v')} \, In\left(\frac{(1+v')(1-v)}{(1-v')(1+v)}\right), \quad |v, v'| > 1 \qquad (2.14)$$

The generating function of the product $Q_n(v')Q_n(v)$ may also obtain using (2.11). To do this, the reader should do the following: first multiply (2.10) by $1/2 \ (v'-x)$ and then integrate over x in the interval $[-1,1]$ and multiply by $1/\sqrt{1-2yt+t^2}$ and then integrate over y in the interval $[-1,1]$. The result is

$$\sum_{n=0}^{\infty} t^n Q_n(v') Q_n(v) = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} \frac{dxdy}{(v'-x)\sqrt{1-2yt+t^2}\sqrt{v^2+x^2+y^2-2vxy-1}},$$

(2.15)

$$-1 \le t \le 1, |v', v| > 1.$$

The integration in (2.15) is a quite laborious work, so we do not pay attention to it. Meanwhile, the complete elliptic of the first kind mentioned in (2.6) is defined as

$$K(k) = \int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{1-k^2u^2}} \tag{2.16}$$

With restriction $0 \le k \le 1$. The incomplete elliptic integral of the first kind mentioned in (2.12) is defined as, with restriction $0 \le z, k < 1$.

$$F(z,k) = \int_0^z \frac{du}{\sqrt{1-u^2}\sqrt{1-k^2u^2}} \tag{2.17}$$

III.MAIN RESULTS

One –dimensional steady-state transport equation for one-energy group is given as

$$x\frac{d\psi(z,x)}{dz} + \sigma_T\psi(z,x) = \int_{-1}^1\int_0^{2\pi} \sigma_S(\mu_0)\psi(z,y)d\varphi' dy + Q/2, \qquad -L \le z \le L, -1 \le x \le 1. \tag{3.1}$$

Where μ_0 is the cosine of the scattering angle, $\sigma_S(\mu_0)$ is the neutron scattering function or kernel, σ_T is the total cross section, $\psi(z,x)$ is the angular flux of neutrons and Q is the external neutron source. Simply we say that $\sigma_S(\mu_0)$ describes the probability that a neutron scatters from an incident directions $\widehat{\Omega}$ to a final direction $\widehat{\Omega'}$. Hence, $\widehat{\Omega}$ and $\widehat{\Omega'}$ are the unit vectors determining the direction of neutrons, $\widehat{\Omega}$ and $\widehat{\Omega'}$ are defined in (2.1) & (2.2)

Neutron scattering functions for one-energy group is given, in terms of Legendre polynomials, by

$$\sigma_S(\mu_0) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}\sigma_{Sn}P_n(\mu_0,), -1 \le \mu_0 \le 1, \tag{3.2}$$

Where $\sigma_{Sn}'s$ are the expansion coefficients. For all reactor calculations, using only first two terms of (3.2) became a traditional behavior. If only the first term of (3.2) is present and remaining terms are zero ($\sigma_{Sn} = 0, n \ge 1$) then system is said to be isotropic. If only the first two terms of (3.2) are present and remaining terms are zero ($\sigma_{Sn} = 0, n \ge 2$) then system is said to be linear anisotropic. Of course, to get an exact solution of (3.1) all the terms in (3.2) should be used, though terms after the second one are negligibly small. To do this we need an analytical expression for the scattering function $\sigma_S(\mu_0)$. To overcome this difficulty we need to define a new scattering kernel should contain the generating function of Legendre polynomials of first kind, that is, the new scattering kernel should be in the form:

$$\sigma_S(\mu_0) = \frac{\sigma_S}{4\pi\sqrt{1-2\mu_0 t + t^2}}, |\mu_0| \le 1, \qquad |t| \le 1, \tag{3.3}$$

Where $\sigma_S$is any non-negative coefficient and 4π is inserted for convenience.

If we integrate (3.3) over φ' in the interval $[0,2\pi]$ we obtain the scattering kernel as

$$\int_0^{2\pi} \sigma_S(\mu_0)d\varphi' = \sigma_S(x,t,y) = \frac{\sigma_S}{\pi}\frac{K(k)}{\sqrt{a-b}}, -1 \le x,y,t \le 1. \tag{3.4}$$

Where $K(k)$ is the complete elliptic integral of the first kind defined in (2.16). If we expand (3.4) in terms of t, we find the following:

$$\sigma_S(x,y,t) = \frac{\sigma_s}{2}\sum_{n=0}^{\infty} t^n P_n(x)P_n(y), \qquad -1 \le x,y,t \le 1. \tag{3.5}$$

Using the addition theorem of Legendre polynomials in (3.2) and then integrating over φ' in the interval $[0,2\pi]$ and equating this result to (3.5), we obtain the following result:

$$\sigma_{Sn} = \sigma_S\frac{t^n}{2n+1}, \quad n = 0,1,\dots,\infty, |t| \le 1. \tag{3.6}$$

Using (3.2) in (3.1) to get an appropriate solution of (3.1) is a traditional way. Using the fact that $\psi(z,x)$ has an azimuthal symmetry, we obtain

$$x\frac{d\psi(z,x)}{dz} + \sigma_t\psi(z,x) - \frac{\sigma_S}{\pi}\int_{-1}^{1}\frac{uK(2u)\psi(z,y)}{\sqrt{t(1-x^2)(1-y^2)}}\,dy = Q/2, \qquad (3.7)$$

Where $u = \sqrt{t(1-x^2)(1-y^2)/(1-2(xy-\sqrt{(1-x^2)(1-y^2)})\ t + t^2}$ and the particular solution of (3.7) is a constant, that is $\psi^P(z,x) = A$, where A is a constant to be determined. Inserting $\psi^P(z,x) = A$ in (3.7) and using the value of the integration

$$\int_{-1}^{1}\frac{uK(2u)dx}{\sqrt{t(1-x^2)(1-y^2)}} = \int_{-1}^{1}\frac{uK(2u)dy}{\sqrt{t(1-x^2)(1-y^2)}} = \pi \qquad (3.8)$$

In (3.7) we find the particular solution as

$$\psi^P(z,x,t) = A = \frac{Q}{2(\sigma_T - \sigma_S)}, -L \le z \le L, -1 \le x, t \le 1. \qquad (3.9)$$

For the solution of homogeneous part of (3.7) we suggest the following *ansatz*

$$\psi^P(z,x,v,t) = H(x,v,t)\exp(-\sigma_T z/v), -L \le z \le L, -1 \le x, t \le 1, |v| > 1. \qquad (3.10)$$

Here, $H(x,v,t)$ is the angular part of the angular flux to be determined. Inserting (3.10) in the homogeneous part of (3.7) and arranging the resulting equation we find

$$H(x,v,t) = \frac{vc_0}{\pi(v-x)}\int_{-1}^{1}\frac{K(k)}{\sqrt{a-b}}H(y,v,t)dy, \qquad c_0 = \sigma_S/\sigma_T, -1 \le x, t \le 1, \qquad |v| > 1$$

(3.11)

Where a, b and k are defined before.

Fredholm integral equation of the second kind is defined as

$$\phi(x) = f(x) + \lambda\int_{\alpha}^{\beta}K(x,y)\phi(y)dy. \qquad (3.12)$$

If we compare (3.11) with (3.12), we draw the conclusion that (3.11) is a Fredholm integral equation of second kind with $\lambda = 1, \alpha = -1, \beta = 1$ and $f(x) = 0$

In this paper, we do not make effort to solve the integral Eq. (3.11), but for the solution we present some hints. For instance, if we set $t = 0$ in (3.11) we find a quite simple integral equation:

$$H(x,v) = \frac{vc_0}{2(v-x)}\int_{-1}^{1}H(y,v)dy, \quad |x| \le 1, \quad |v| > 1 \qquad (3.13)$$

This integral equation appears in the reactor physics when the medium, in which the neutrons are diffusing, is isotropic. Here v and $H(x,v)$ are known as discrete eigenvalues and eigenfunctions, respectively. Solution of (3.13) becomes quite simple if we choose the normalization condition as

$$\int_{-1}^{1}H(y,v)dy = 1 \qquad (3.14)$$

And using this in (3.13) we find

$$H(x,v) = \frac{vc_0}{2(v-x)}, |x| \le 1, \quad |v| > 1 \qquad (3.15)$$

And using the eigenfunction in (3.14) we find the eigenvalue equation for the isotropic scattering as

$$\frac{vc_0}{2}\, In\left(\frac{v+1}{v-1}\right) = 1, \quad |v| > 1 \qquad (3.16)$$

To derive more general eigenvalue equation we can use (3.11). By using (2.6) in (3.11) we obtain

$$H(x,v,t) = \frac{vc_0}{2(v-x)}\sum_{n=0}^{\infty}t^nP_n(x)\int_{-1}^{1}P_n(y)H(y,v,t)dy, \quad |x,t| \le 1, \quad |v| > 1 \qquad (3.17)$$

Multiplying both sides of (3.17) by $(v-x)P_m(x)$ and integrating over x in the interval $[-1,1]$ we obtain

$$(n+1)\alpha_{n+1}(v,t) - v(2n+1)\alpha_n(v,t) + n\alpha_{n-1}(v,t) = -vc_0t^n\alpha_n(v,t) \qquad (3.18)$$

Where we define a new function $\alpha_n(v,t)$ as

$$\alpha_n(v,t) = \int_{-1}^{1}P_n(y)H(y,v,t)dy, \qquad (3.19)$$

For which the case $n = 0$ is the normalization condition given by

$$\alpha_0(v,t) = 1 = \int_{-1}^{1} H(y,v,t)dy, \tag{3.20}$$

A few $\alpha_n(v,t)$ can be obtain, using (3.18) as

$$\alpha_0(v,t) = 1 \tag{3.21}$$

$$\alpha_1(v,t) = v(1-c_0) \tag{3.22}$$

$$\alpha_2(v,t) = \frac{v^2}{3}(3-c_0t)\,(1-c_0) - \frac{1}{2} \tag{3.23}$$

To obtain (3.18) we used the orthogonality relation of Legendre polynomial defined as

$$\int_{-1}^{1} P_n\,(x)P_m(x)dx = \frac{2\delta_{nm}}{2n+1}$$

IV. CONCLUSIONS

This paper involves the application of useful properties of Legendre polynomial to the neutron transport equation. In order to obtain the appropriate solution of neutron transport equation, the suitable scattering function of the neutron must be defined. All these coefficients i.e. $\sigma_{Sn}{}'s$ are connected to one parameter with the help of equation (3.6). Hence, it is possible to calculate numerically the coefficients of $\sigma_{Sn}{}'s$. By defining a new scattering Kernel we can demonstrate the theoretical scheme for the solution of one group and one-dimensional neutron transport equation.

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