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Approximate Solution of One-dimensional Heat Conduction Problem

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***Abstract**:In this paper, one-dimensional heat equation subject to both Neumann and Dirichlet initial boundary conditions is presented and a Homotopy Perturbation Method is utilized for solving the problem. Homotopy Perturbation Method provides continuous solution in contrast to finite difference method, which only provides discrete approximations. It is found that this method is a powerful mathematical tool and can be applied to a large class of linear and non linear problem in different fields of science, engineering and technology.*
***Keywords**:Differential operator, Directional derivative, Convergence, Heat conduction, Diffusion coefficient*

I. INTRODUCTION

Analytical methods have gained the interest of researchers for finding approximate solutions to partial differential equations. This interest was driven by the needs from applications both in science and technology. There has been a growing interest in the new analytical techniques for linear and non-linear initial value boundary problems. The widely applied techniques are perturbation method which is called the homotopy perturbation methods. He has proposed a new perturbation technique which is called the Homotopy Perturbation Method. Homotopy Perturbation Methodhas gained reputation as being powerful tool for solving linear or non linear partial differential equations. Homotopy Perturbation Method was applied to solve initial boundary value problems which are governed by the non linear ordinary (Partial) differential equations; the results show that this method is efficient and simple. Thus, the main goal of his work is to apply homotopy perturbation method for solving one dimensional heat conduction problem with Dirichlet and Neumann boundary conditions. The combined results are more accurate than others.

II. NOTATIONS AND DEFINITIONS

The general form of equation is gives as:

$$u_t = \alpha u_{xx} + f(x,t), 0 < x < l, t > 0 \tag{2.1}$$

Subject to the initial condition:

$$u(x,0) = u_0(x), 0 < x < a \tag{2.2}$$

And the boundary conditions

$$u(0,t) = g_0(t), u(l,t) = g_1(t), t > 0 \tag{2.3}$$

$$u_x(0,t) = g_2(t), u_x(l,t) = g_3(t), t > 0 \tag{2.4}$$

Where the diffusion coefficient α is positive, $u(x,t)$ represents the temperature at point (x,t) and $f(x,t)$, $g_0(t), g_1(t), g_2(t), g_3(t)$ are sufficiently smooth known functions.

To illustrate the basic ideas, let X, and Y be two topological spaces. If f and g are continuous maps of the spaces X into Y, it is said that f is homotopic to g, if there is continuous map $F: X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for each $x \in X$, then the map is called homotopy between f and g.

We consider the following nonlinear partial differential equation:

$$A(u) - f(r) = 0, r \epsilon \Omega \tag{2.5}$$

Subject to the boundary conditions

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, r \epsilon \Gamma \tag{2.6}$$

Where A is a general differential equation operator. f is a known analytical function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial \eta}$ denotes directional derivative in outward normal direction to Ω. The operator A, generally divided into two parts, L and N where L is linear, while N is nonlinear. Using A=L+N, eq. (2.5) can be rewritten as follows

$$L(v) + N(v) - f(r) = 0 \tag{2.7}$$

By the homotopy technique, we construct a homotopy defined as

$H(v,p): \Omega \times [0,1] \to R$ (2.8)

Which satisfies

$H(v,p) = (1-p)\big(L(v) - L(u_0)\big) + p\big(A(v) - f(r)\big),$ (2.9)

$p \in [0,1], r \in \Omega$

Or

$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\big(N(v) - f(r)\big) = 0$ (2.10)

$p \in [0,1], r \in \Omega$

Where $p \in [0,1]$ is an embedding parameter, v_0 is an initial approximation of equation (2.5), which satisfies the boundary conditions. It follows from equation (2.10) that:

$H(v,0) = L(v) - L(u_0) = 0$ (2.11)

$H(v,1) = A(v) - f(r) = 0$ (2.12)

The changing process of p from 0 to 1 monotonically is trivial problem. $H(v,0) = L(v) - L(u_0) = 0$is continuously transformed to the original problem

$H(v,1) = A(v) - f(r) = 0.$ (2.13)

In topology, this process is known as continuous deformation. $L(v) - L(u_0)$and$A(v) - f(r)$ are called homotopic. We use the embedding parameter p as a small parameter, and assume that the solution of equation (2.11) can be written as power series of p:

$p = p^0 v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots + p^n v_n + \cdots$

(2.14)

Setting $p = 1$ we obtain the approximate solution of equation (2.5) as:

$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots + v_n + \cdots$

(2.15)

The series equation (2.15) is convergent for most of the cases, but the rate of the convergence depends on the nonlinear operator $N(v)$.

III.MAIN RESULTS

A. We consider the problem

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 \le x \le 1, t > 0$ (3.1)

With the initial condition:

$u(x,0) = \sin(\pi x),$

And the boundary condition

$u(0,t) = 0, u(1,t) = 0$ (3.2)

For the solving this problem, we construct the Homotopy Perturbation Method as follows:

$$H(v,p) = (1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial u}{\partial t} - \frac{\partial^2 v}{\partial x^2}\right) = 0$$

(3.3)

The component v, of (2.15) are obtained as follows:

$\frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, v_0 = u(x,0) = \sin(\pi x)$ (3.4)

$\frac{\partial v_1}{\partial t} - \frac{\partial^2 v_0}{\partial t} = 0, v_1(x,0) = 0$ (3.5)

$$\frac{\partial^2 v_0}{\partial x^2} = -\pi^2 \sin\,(\pi x)$$

$$\frac{\partial v_1}{\partial t} = -\pi^2 \sin\,(\pi x)$$

Hence

$v_1 = -\pi^2 \sin(\pi x)t$ (3.6)

$\frac{\partial v_2}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} = 0, v_2(x,0) = 0$ (3.7)

$\frac{\partial^2 v_1}{\partial t^2} = 12t^2 + 24t, \frac{\partial v_1}{\partial t} - \frac{\partial^2 v_0}{\partial x^2} = 0$

$\frac{\partial v_2}{\partial t} = \pi^4 \sin(\pi x)\, t$

$v_2 = -\pi^4 \sin(\pi x) \frac{t^2}{2!}$ (3.8)

Fr the next component:

$\frac{\partial v_3}{\partial t} - \frac{\partial^2 v_2}{\partial x^2} = 0, v_3(x,0) = 0$

$= -\pi^6 \sin(\pi x) \frac{t^3}{3!}$ (3.9)

And so on, we obtain the approximate solution as follows:

$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots + v_n + \cdots$

And this leads to the following solution

$u(x,t) = \sin(\pi x)\, e^{-\pi^2 t}$ (3.10)

We can immediately observe that this solution is exact.

B. Consider the following nonlinear reaction-diffusion equation

$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \le x \le 1, t > 0$ (3.11)

Subject to the initial condition

$u(x,0) = \cos(\pi x)$ (3.12)

And the boundary conditions

$\frac{\partial u(,t)}{\partial x} = 0, \frac{\partial u(1,t)}{\partial x} = 0$ (3.13)

Solving the equation (3.11) with the initial condition (3.12) yields:

$\frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, v_0 = u_0 = \cos(\pi x)$

$\frac{\partial v_1}{\partial t} - \frac{\partial^2 v_0}{\partial x^2} = 0, v_1 = -\pi^2 \cos(\pi x)t, v_1(x,0) = 0$

$\frac{\partial v_2}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} = 0, v_2 = -\pi^4 \cos(\pi x) \frac{t^n}{2!},$

And we can deduce the remaining components as :

$, \dots , v_n = (1)^n \pi^{2n} \cos(\pi x) \frac{t^n}{n!},$

(3.14)

Using equation in the above, we get:

$u(x,t) = \cos(\pi x)\, (1 - \frac{\pi^2 t}{1!} + \frac{(\pi^2 t)^2}{2!} - \frac{(\pi^2 t)^3}{3!} + \cdots$

And finally the approximate solution is obtained as

$u(x,t) = e^{-\pi^2 t} \cos(\pi x)$ (3.15)

IV. CONSIDER THE FOLLOWING PROBLEM:

$u_t =$
$u_{xx}(\pi^2 - 1)e^{-t}\cos(\pi x) + 4x - 2,$ (3.16)
$0 \le x \le 1, t > 0$
With th initial condition
$u(x,0) = \cos(\pi x) + x^2$ (3.17)
And the boundary conditions:
$u(0,t) = e^{-t}, u(1,t) = -e^{-t} + 4t + 1$
According to the Homotopy Perturbation Method, we have

$$H(v,p) = (1-p)\left(\frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} +\right) p\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - f\right) = 0 \quad (3.18)$$

Where $f = (\pi^2 - 1)e^{-t} + 4x - 2$

By equating the terms with the identical powers of π,yields
$p^0: \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \frac{\partial v_0}{\partial t} = 0, v_0 = \cos(\pi x) + x^2$ (3.19)

$p^0: \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \frac{\partial v_0}{\partial t} = 0, v_0 = \cos(\pi x) + x^2$
$p^1: \frac{\partial v_1}{\partial t} - \frac{\partial^2 v_0}{\partial x^2} = 0, v_1(x,0) = 0$
$\frac{\partial v_1}{\partial t} = 4x + \cos(\pi x)(-\pi^2 + (\pi^2 1)e^{-t})$
$v_1 = 4xt + \cos(\pi x)(-\pi^2 t + (\pi^2 - 1)(1 - e^{-t}))$
$p^2: \frac{\partial v_2}{\partial t} - \frac{\partial {v_1}^2}{\partial x^2} = 0, v_2(x,0) = 0.$
$\frac{\partial v_2}{\partial t} = \cos(\pi, 0)\ (\pi^4 t - \pi^2(\pi^2 - 1)(1 - e^{-t}))$
Continuing lie-wise we get:
$v_2 = \cos(\pi x)\ ((\pi^4 - \pi^2)\left(1 - \frac{t}{1!} - e^{-t}\right) + \frac{(\pi^2 t)^2}{2!})$

$v_3 = \cos(\pi x)\ ((\pi^6 - \pi^4)\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - e^{-t}\right)$
$+ \frac{(\pi^2 t)^3}{3!})$
$v_4 = \cos(\pi x)\ ((\pi^8 - \pi^6)\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} - e^{-t}\right)$
$+ \frac{(\pi^2 t)^4}{4!})$
And so on then we have:

$$u_{5hpm} = x^2 + 4xt + \cos(\pi x)\left[(\pi^8\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} - e^{-t}\right) + e^{-t}\right] \quad (3.20)$$

From this result we deduce that the series solution converges to the exact one:

$u(x,t) = x^2 + 4xt + \cos(\pi x)\, e^{-t}$

V. CONSIDER THE NON-HOMOGENEOUS HEAT EQUATION WITH NON LINEAR-HOMOGENEOUS NEUMANN BOUNDARY CONDITIONS:

$$u_t = u_{xx} + \left(\frac{\pi^2}{2}\right)e^{\frac{-\pi^2}{2}t}\cos(\pi x) + x - 2 \quad (3.21)$$

$0 \le x \le 1, t > 0$

$u(x,0) = \cos(\pi x) + x^2, u_x(0,t) = t$

$u_x(1,t) = 2 + t$

The theoretical solution is:

$u(x,t) = x^2 + xt + e^{\frac{-\pi^2}{2}t}\cos(\pi x)$

According to Homotopy Perturbation Method, we get the components of

$$v_{0t} - u_{0t} = 0, v_0 = \cos(\pi x) + x^2 \quad (3.22)$$

$$v_{1t} = v_{0xx} + \frac{\pi^2}{2}e^{\frac{-\pi^2}{2}t}\cos(\pi x) + x - 2, v_1(x,0) = 0$$

$$v_{1t} = x + \cos(\pi x)\left(-\pi^2 + \frac{\pi^2}{2}e^{\frac{-\pi^2}{2}t}\right)$$

$$v_1 = xt + \cos(\pi x)\left(1 - \pi^2 t - e^{\frac{-\pi^2}{2}t}\right) \quad (3.23)$$

$$v_{2t} = v_{1xx} = \cos(\pi x)\left(-\pi^2 + \pi^4 t + \pi^2 e^{\frac{-\pi^2}{2}t}\right)$$

$$v_2 = \cos(\pi x)\,(2 - \pi^2 t + \frac{(\pi^2 t)^2}{2!} - 2e^{\frac{-\pi^2}{2}t}$$

$$v_{3t} = v_{2xx} = \cos(\pi x)\,(-2\pi^2 + \pi^4 t - \frac{\pi^6 t^2}{2!}$$
$$+2\pi^2 e^{\frac{-\pi^2}{2}t})$$

$$v_3 = \cos(\pi x)\,(4 - 2\pi^2 t + \frac{(\pi^2 t)^2}{2!} - \frac{(\pi^2 t)^3}{3!}$$
$$-4e^{\frac{-\pi^2}{2}t}) \quad (3.24)$$

$$v_{4t} = v_{3xx} = \cos(\pi x)\,(-4\pi^2 + 2\pi^4 t - \pi^2\frac{(\pi^2 t)^2}{2!}$$
$$+ \pi^2\frac{(\pi^2 t)^3}{3!} + 4\pi^2 e^{\frac{-\pi^2}{2}t}$$

$$v_4 = \cos(\pi x)\,8 - 4(\pi^2 t) + (\pi^2 t)^2 - \frac{(\pi^2 t)^3}{3!}$$
$$+ \frac{(\pi^2 t)^4}{4!} - 8e^{\frac{-\pi^2}{2}t})$$

And so on, we obtain the approximate solution as follows

$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + v_4 \ldots$

Or

$$u(x,t) = x^2 + xt + \cos(\pi x)\,e^{\frac{-\pi^2}{2}t}$$
$$+15\left\{(1 - \frac{\left(\frac{\pi^2}{2}t\right)}{1!} + \frac{\left(\frac{\pi^2}{2}t\right)^2}{2!} - \frac{\left(\frac{\pi^2}{2}t\right)^3}{3!} + \frac{\left(\frac{\pi^2}{2}t\right)^4}{4!} -. + \cdots) - e^{\frac{-\pi^2}{2}t}\right\}$$

And this leads to the following solution

$$u(x,t) = x^2 + xt + \cos(\pi x) - e^{\frac{-\pi^2}{2}t}$$

This solution coincides with the exact one.

VI. CONCLUSIONS

This paper has constructed an approximate solution of the heat conduction problem with Dirichlet and Neumann boundary conditions with the use of Homotopy perturbation method. The problem solved using Homotopy Perturbation Method gave better results than those using finite difference schemes. The case studies are in agreement with the exact solution. These results do not involve linearization, discretization, transformation or restrictive assumptions. These results depict the stability and convergence of the method.

VII. ACKNOWLEDGMENT

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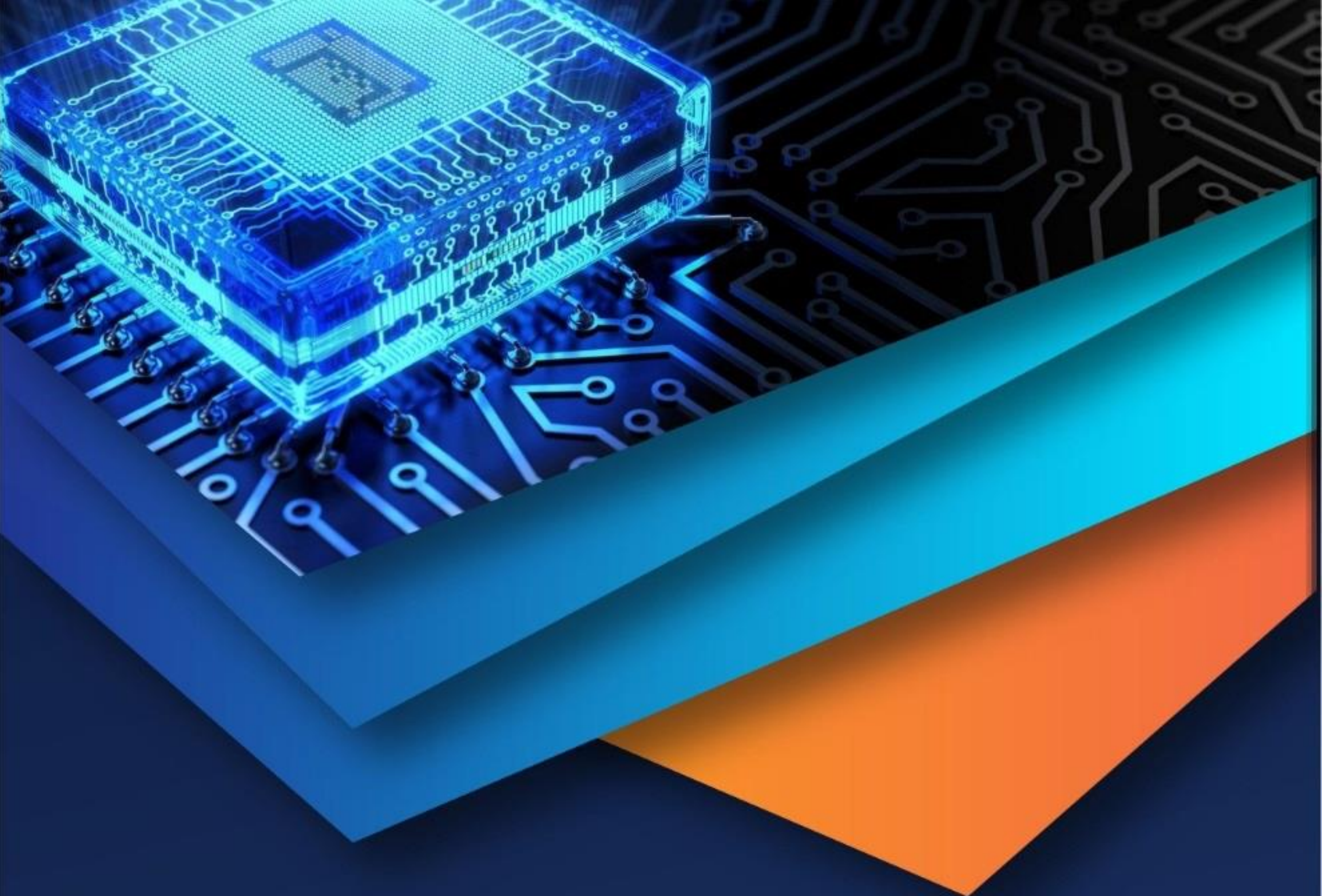

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