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Boundary Value Problem of Fractional Differential Equation

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Abstract: ***In this paper, we prove the existence of the solution for the boundary value problem of fractional differential equations of order $q \in (2,3]$.The Krasnoselskii's fixed point theorem is applied to establish the results.***
Keywords: ***Fractional differential equation, Krasnosels kii's fixed point theorem, Boundary value problem, Positive Solution, Gamma functions***

I. INTRODUCTION

Fractional differential equations are the generalization of ordinary equation to arbitrary non-integer order, and have received more and more interest due to their wide applications in various branch of science & engineering, such as physics, chemistry, biophysics, capacitor theory, blood flow phenomena, electrical circuits, control theory, etc , also recent investigations have demonstrated that the dynamics of many systems are described more accurately by using fractional differential equations.Nickolai was concerned with the nonlinear differential equation of fractional order

$$D_{0+}^{q} u(t) = f\big(t, u(t), u'(t)\big) \quad a.e.\, t \in (0,1),$$

Where $D_{0+}^{q}$is Riemann-Liouville (R-L) fractional order derivatives, subject to the boundary conditions $u(0) = u(1) = 0$.
Zhang has given the existence of positive solution to the equations

$$\begin{cases} {}^{c}D^{q}u(t) + f(t, u(t)) = 0, 0 < t < 1, \\ u(0) + u'(0) = u(1) + u'(1) = 0, \end{cases}$$

By the use of classical fixed point theorems, where ${}^{c}D^{q}$ denotes Caputo fractional derivative with$1 < q \le 2$. Chen considered the existence of three positive solutions to three-point boundary value problem of the following fractional differential equation

$$\begin{cases} D_{0+}^{q} u(t) + f(t, u(t)) = 0, 0 < t < 1 \\ u(0) = 0, D_{0+}^{p} u(t)\big|_{t=1} = \alpha D_{0+}^{p} u(t)\big|_{t=\xi} \end{cases}$$

Where $1 < q \le 2, 0 < p < 1, 1 + p \le q$, and D_{0+}^{a} is the R-L fractional order derivative. The multiplicity results of positive solutions to the equations are obtained by using the well-known Leggett-Williams fixed-point theorem on convex cone, we study the existence of positive solution to two point BVP of nonlinear fractional equation.

$$\begin{cases} D_{0+}^{q} u(t) + \lambda f\big(t, u(t)\big) = 0, 0 < t < 1, \\ u(0) = D_{0+}^{p} u(t)\big|_{t=0} = D_{0+}^{p} u(t)\big|_{t=1} = 0 \end{cases} \tag{1.1}$$

Where$q, p \in R, 2 < q \le 3, 1 < p \le 2, 1 + p \le q, D_{0+}^{q}$ is the R-L fractional order derivative, and$f \in C\big([0,1] \times [0,\infty), [0,\infty)\big)$, $\lambda > 0$.

II. NOTATIONS AND DEFINITIONS

Definition 1. The R-L fractional integrals $I_{0+}^{p} f$ *of order* $p \in R (p > 0)$ defined by

$$I_{0+}^{p} f(x) := \frac{1}{\Gamma(p)} \int_{0}^{x} \frac{f(t)dt}{(x-t)^{1-p}}, (x > 0).$$

Here Γ (p) is the Gamma function.
Definition 2. The R-L fractional derivatives $D_{0+}^{p} f$ order $p \in R$ $(p > 0)$ is defined by

$$D_{0+}^{p} f(x) = \left(\frac{d}{dx}\right)^{n} I_{0+}^{n-p} f(x)$$

$$= \frac{1}{\Gamma(n-p)}\left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)dt}{(x-t)^{p-n+1}}, \quad (n = 1[p]+1, x > 0),$$

Where p means the integral part of p.

III.MAIN RESULTS

Lemma 1.If $q_1 > q_2 > 0$, then, for $f(x) \in L_p(0,1), (1 \le p \le \infty)$ the relations
$D_{0+}^{q_2} I_{0+}^{q_1} f(x) = I_{0+}^{q_1-q_2} f(x), I_{0+}^{q_1} I_{0+}^{q_2} f(x) = I_{0+}^{q_1+q_2} f(x)andD_{0+}^{q_1} I_{0+}^{q_1} f(x) = f(x)$
holdae. on [0,1].

Lemma 2. Let $q > 0, n = [q]+1, f(x) \in L_1(0,1)$, then the equality

$$I_{0+}^q D_{0+}^q f(x) = f(x) + \sum_{i=1}^n C_i t^{q-n}.$$

Lemma 3.Let $y \in C[0,1], 2 < q \le 3, 1 < p \le 2, 1+p \le q$, then the problem

$$D_{0+}^q u(t) + y(t) = 0, 0 < t < 1, \tag{3.1}$$

subject to the boundary conditions

$$u(0) = D_{0+}^p u(t)\Big|_{t=0} = D_{0+}^p u(t)\Big|_{t=1} = 0 \tag{3.2}$$

has the unique solution $u(t) = \int_0^1 G(t,s)ds$, where

$$G(t,s) = \frac{1}{\Gamma(q)}\begin{cases} t^{q-1}(1-s)^{q-p-1} - (t-s)^{q-1}, & 0 \le s \le t \le 1 \\ t^{q-1}(1-s)^{q-p-1} & 0 \le t \le s \le 1 \end{cases}$$

And that $G(t,s)$ has the following properties

(i) $G(t,s) \in C([0,1]\times[0,1])$ and $G(t,s) > 0$ for $t,s\ (0,1)$, and $\max_{0\le t\le 1} G(t,s) = G(s,s), s \in (0,1)$.

(ii) There exists a positive function $\varphi \in C\left((0,1)\times(\tau,\infty)\right)$ such that

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G(t,s) = \varphi(s)\tilde{G}(s,s) \ge \inf_{0<s<1}\varphi(s) \max_{0\le t\le 1} G(t,s) = \tau G(s,s),$$

Where

$$\tilde{G}(s,s) = \frac{s^{q-p}(1-s)^{p-q-1}}{\Gamma(q)}, s,\tau \in (0,1), \tau = \inf_{0<s<1}\varphi(s).$$

Proof. Applying the operator I_{0+}^q to both sides of the equation (1.1), and using Lemma 2, we have

$$u(t) = -I_{0+}^q y(t) + c_1 t^{q-1} + c_2 t^{q-2} + c_3 t^{q-3} \tag{3.3}$$

In view of the boundary condition $u(0) = 0$, we find that $C_3 = 0$ hence
$u(t) = -I_{0+}^q y(t) + C_1 t^{q-1} + C_2 t^{q-2}$,
then , noting the relation $D_{0+}^{q_2} I_{0+}^{q_1} f(x) = I_{0+}^{q_1-q_2} f(x)$ in Lemma 1, we obtain

$$D_{0+}^p u(t) = -I_{0+}^{q-p}\, y(t) + C_1 \frac{\Gamma(q)}{\Gamma(q-p)} t^{q-1-p} + C_2 \frac{\Gamma(q-1)}{\Gamma(q-p-1)} t^{q-p-2},$$

In accordance with the equation (3.1), we can calculate out that

$$C_1 = \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-p-1} y(s)ds, c_2 = 0.$$

Substituting the vlues of C_1, C_2 and C_3 in (3.2) we have

$$u(t) = -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} y(s)ds + \frac{t^{q-1}}{\Gamma(q)}\int_0^1 (1-s)^{q-p-1} y(s)ds$$

$$= \frac{1}{\Gamma(q)}\left\{\int_0^t [t^{q-1}(1-s)^{q-p-1}y - (t-s)^{q-1}]\, y(s)ds + \int_t^1 [t^{q-1}(1-s)^{q-p-1}]\, y(s)ds\right\}$$

$$= \int_0^1 G(t,s)y(s)ds$$

Next we prove the properties of $G(t,s)$.
For a given $s \in (0,1)$, $G(s,t)$is the decreasing with respect to t for $s \le t$ while increasing for $t \le s$. Thus, we have

$$\max_{0\le t\le 1} G(t,s) = G(s,s) = \frac{s^{q-1}(1-s)^{q-p-1}}{\Gamma(q)} \le \frac{s^{q-p}(1-s)^{q-p-1}}{\Gamma(q)} = \widetilde{G}(s,s),$$

for $s \in (0,1)$. Then we set

$$g_1(t,s) = \frac{t^{q-1}(1-s)^{q-p-1} - (t-s)^{q-1}}{\Gamma(q)}, g_2(t,s) = \frac{t^{q-1}(1-s)^{q-p-1}}{\Gamma(q)},$$

from the two equation above we have

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G(t,s) = \frac{1}{\Gamma(q)}\begin{cases} 0.75^{q-1}(1-s)^{q-p-1} - (0.75-s)^{q-1}, & 0 < s \le r \\ 0.25^{q-1}(1-s)^{q-p-1} & r \le s < 1 \end{cases}$$

where $\frac{1}{4} < r < \frac{3}{4}$ is the unique solution of the equation.

$$0.75^{q-1}(1-s)^{q-p-1} - (0.75-s)^{q-1} = 0.25^{q-1}(1-s)^{q-p-1}$$

Finally, we consider a function $\varphi(s)$ defined by

$$\varphi(s) = \frac{\min_{\frac{1}{4}\le t\le\frac{3}{4}} G(t,s)}{\widetilde{G}(s,s)} = \begin{cases} \dfrac{0.75^{q-1}(1-s)^{q-p-1} - (0.75-s)^{q-1}}{s^{q-p}(1-s)^{q-p-1}}, & 0 < s \le r, \\ \dfrac{0.25^{q-1}}{s^{q-p}}, & 0 \le s < 1. \end{cases}$$

when $p > q-1$ we find from the continuity of $\varphi(s)$ and $\lim_{s\to 0^+} = +\infty$ that there exists $\tilde{r}$ small enough such that $\varphi'(s) < 0$ for $s \in (0, \tilde{r}]$ hence , we set

$$0 < r = \inf_{0<s<1} \varphi(s) = \min\left\{\varphi(\tilde{r}), m, \frac{1}{4^{q-1}}\right\} < 1,$$

here, $m = \min_{\tilde{r}\le s\le r} \varphi(s)$.

when $q = p-1$, we have $\lim_{s\to 0^+} \varphi(s), \frac{4}{3}(q-1)$, then we set

$$0 < \tau = \inf_{0<s<1} \varphi(s) = \min\left\{\inf_{0<s\le r} \varphi(s), \frac{3}{4}(q-1), \frac{1}{4^{q-1}}\right\} < 1.$$

Thus

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G(t,s) \ge \varphi(s)\widetilde{G}(s,s) \ge \inf_{0<s<1} \varphi(s) \max_{0\le t\le 1} G(t,s) = \tau G(s,s).$$

This completes the proof. Therefore the solution $u \in C_{[0,1]}$ of the problem (1.1) can be written by

$$u(t) = \lambda \int_0^1 G(t,s) f(s,u(s)) ds.$$

IV. CONCLUSIONS

The paper proves the existence of the solution for boundary value problem of fractional differential equations of the order $q \in (2,3]$. and three Lemmas are established.

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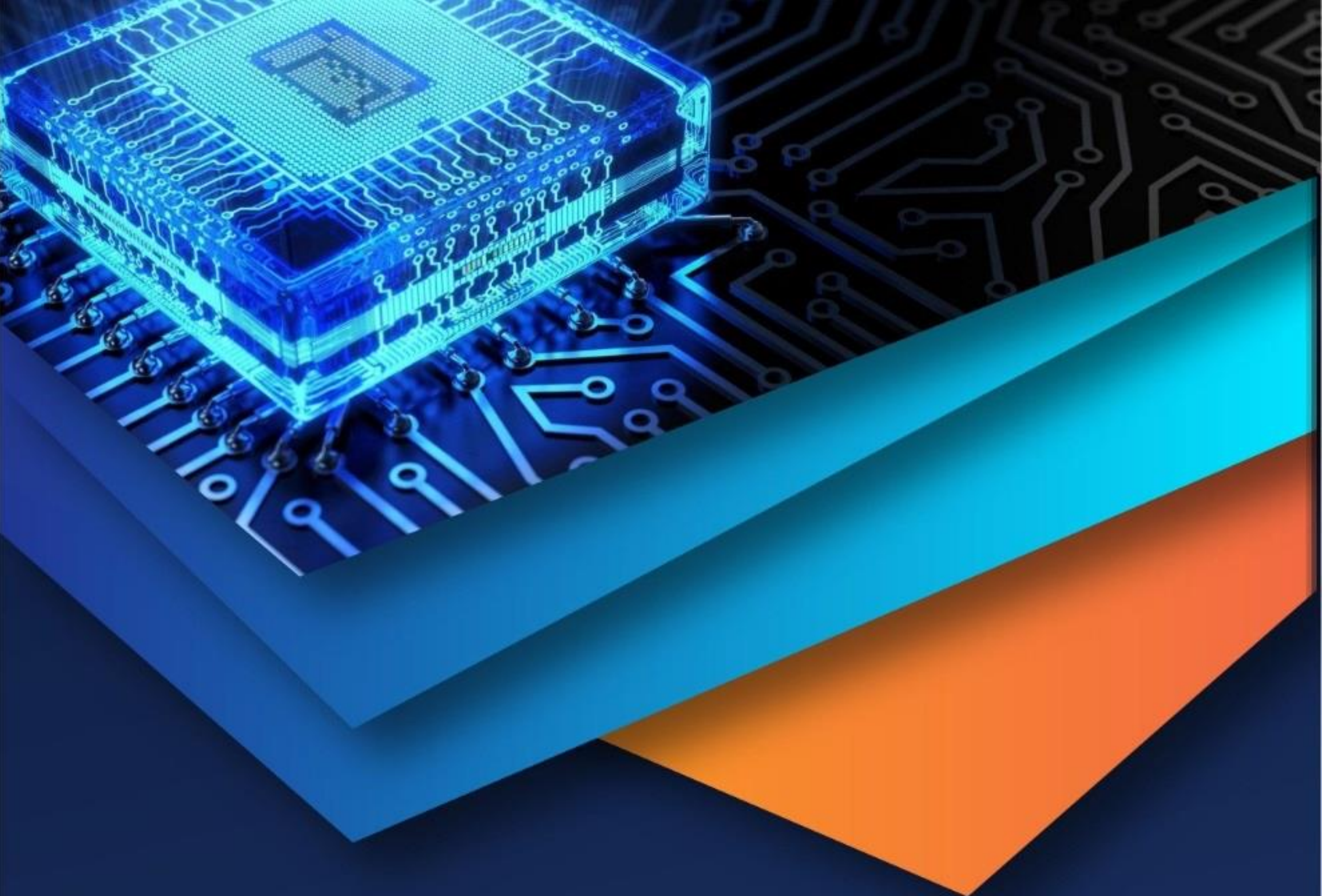

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