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Characterization of Quotient Multi-Fuzzy Subgroup

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Abstract: In this paper we define a new algebraic structure of quotient multi-fuzzy subgroupof a group and some of its properties under homomorphism and anti-homomorphism are investigated. The purpose of this study is to implement the fuzzy set theory and group theory in multi-fuzzy set. Characterization of quotient multi-fuzzy subgroup of a group is given.
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I. INTRODUCTION

L.A.Zadeh [9] introduced the theory of fuzzy set in 1965. Rosenfeld [6] introduced fuzzy subgroups in 1971. P.S.Das[2] studied the inter-relationship between the fuzzy subgroup and its level subgroups in 1981. Mukharjee and Bhattacharya [4] proposed the concept of normal fuzzy subgroups and fuzzy cosets in 1984. S.Sabu and T.V.Ramakrishnan[7, 8] proposed the theory of multi-fuzzy set in terms of multi-dimensional membership functions and investigated some properties of multi-level fuzziness in 2010 and 2011. R.Muthuraj and S.Balamurugan[5] proposed the inter-relationship between the multi-fuzzy subgroup and its level subgroups in 2013.Several researchers N.Ajmal [1], Kumar I.J., Saxena P.K., and Yadav P. [3], etc., are developed the concept of fuzzy normal subgroup and fuzzy quotient group.In this paper we define a new algebraic structure of multi-fuzzy quotient group of a group and also establish some of its related properties.

II. PRELIMINARIES

In this section, we site the fundamental definitions that will be used in the sequel.

A. Definition2.1

Let X be any non-empty set. A fuzzy subset μ of X is $\mu : X \rightarrow [0,1]$.

B. Definition2.2

Let Xbe a non-empty set. A multi-fuzzy set Ain Xis defined as a set of ordered sequences: $A = \{ (x,\mu_1(x), \mu_2(x), ..., \mu_i(x), ...): x\in X\}$, where $\mu_i : X \rightarrow [0,1]$ for all i.

C. Remark 2.3

1) If the sequences of the membership functions have only k-terms (finite number of terms), then k is called the dimension of A
2) The multi-fuzzy membership function $\mu_A$is a function from X to $[0,1]^k$such that for all x in X, $\mu_A(x) = (\mu_1(x), \mu_2(x), ...,\mu_k(x))$
3) For the sake of simplicity, we denote the multi-fuzzy set $A= \{ (x,\mu_1(x), \mu_2(x), ..., \mu_k(x)): x\in X\}$ as $A=(\mu_1, \mu_2, ..., \mu_k)$.

D. Definition 2.4

Let k be a positive integer and let Aand Bbe two multi-fuzzy subsets of a non-empty set X, where $A= (\mu_1, \mu_2, ...,\mu_k)$ and $B= (\nu_1, \nu_2, ..., \nu_k)$, then we have the following relations and operations:

1) $A \subseteq B$ if and only if $\mu_i\leq \nu_i$, for all i = 1, 2, ..., k;
2) A= Bif and only if $\mu_i= \nu_i$, for all i = 1, 2, ..., k;
3) $A\cup B= (\mu_1\cup\nu_1, ..., \mu_k\cup\nu_k) =\{(x, \max(\mu_1(x), \nu_1(x)), ..., \max(\mu_k(x), \nu_k(x))): x\in X\}$;
4) $A\cap B= (\mu_1\cap\nu_1, ..., \mu_k\cap\nu_k) = \{(x, \min(\mu_1(x), \nu_1(x)), ..., \min(\mu_k(x), \nu_k(x))): x\in X\}$;

E. Definition 2.5

Let A= (μ_1, μ_2, ...,μ_k) be a multi-fuzzy subset of X having dimension k and let μ_i'be the fuzzy complement of the ordinary fuzzy set μ_i for each i = 1, 2, ..., k.The complementof a multi-fuzzy set Ais a multi-fuzzy setA^C = (μ_1', μ_2', ...,μ_k') where each $\mu_i' = 1-\mu_i$ for i=1,2,…,k. That is, A^C= {(x,1−μ_1(x), ...,1−μ_k(x)): x∈X}.

F. Definition 2.6

Let μ be a fuzzy set on a group G. Then μ is said to be a fuzzy subgroup of G if

1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ and

2) $\mu(x^{-1}) = \mu(x)$.

G. Definition 2.7

A multi-fuzzy set A of a group G is called a multi-fuzzy subgroup of G iffor allx, y∈G,

1) $A(xy) \geq \min\{A(x), A(y)\}$ and

2) $A(x^{-1}) = A(x)$.

H. Definition 2.8

Let A be a multi-fuzzy subgroup of a group G. For any a∈G, define $(aA)(x) = A(a^{-1}x)$ for all x∈G is called a multi-fuzzy coset of a multi-fuzzy subgroup Aof the group G determined by the element a∈G.

I. Remark 2.9

If a = e in G, then the multi-fuzzy cosetaA = A, where A is a multi-fuzzy subgroup of group G.

III. PROPERTIES OF MULTI-FUZZY QUOTIENT GROUP $\overline{A}$ OF AGROUP G DETERMINED BY A AND K

In this section, we discuss some of the properties of multi-fuzzy quotient group $\overline{A}$ of a group G determined by A and K.

A. Theorem 3.1

Let A be a normal multi-fuzzy subgroup of a group G with identity e. Let K = { x∈G / A(x) = A(e) }. Consider $\overline{A} = (\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots, \overline{A}_k)$ which is defined by

$\overline{A}(xK) = \sup_{k\in K} A(xk)$, for all x∈G , where each $\overline{A}_i : G/K \to [0,1]$. Then

1) K is a normal subgroup of G

2) The multi-fuzzy set $\overline{A}$ is well-defined

3) $\overline{A}$ is a multi-fuzzy subgroup of G/K .

4) *Proof:* Given A is a normal multi-fuzzy subgroup of G and

a) K = { x∈G / A(x) = A(e) }. Let x∈G and y∈K. Then A(y) = A(e).

Since A is a normal multi-fuzzy subgroup of G, $A(xyx^{-1}) = A(y) = A(e)$.Hence,$xyx^{-1}$∈K. Hence, K = { x∈G / A(x) = A(e) } is a normal subgroup of G.

b) Consider $\overline{A} = (\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots, \overline{A}_k)$ which is defined by $\overline{A}(xK) = \sup_{k\in K} A(xk)$, for all x∈G , where each $\overline{A}_i : G/K \to$ [0,1].

Let xK = yK for some x, y∈G.

Thenxy^{-1}∈ K.

That is, $A(xy^{-1}) = A(e)$.

That is, $\overline{A}(xK) = \overline{A}(yK)$.

Hence, the map $\overline{A}$ is well-defined.

Now, $\overline{A}(xKyK) = \overline{A}(xyK)$

$= \sup_{k\in K} A(xyk)$, for all x, y$\in$G.

$\geq \sup_{k_1, k_2\in K} \{ \min\{(A(xk_1) , A(yk_2)\} \}$.

$\geq \min\{ \sup_{k_1\in K} A(xk_1) , \sup_{k_2\in K} A(yk_2) \}$.

$\geq \min\{ \overline{A}(xK) , \overline{A}(yK) \}$.

$\overline{A}(xKyK) \geq \min\{ \overline{A}(xK) , \overline{A}(yK) \}$.

$\overline{A}((xK)^{-1}) = \overline{A}(x^{-1}K)$

$= \sup_{k\in K} A(x^{-1}k)$, for all x$\in$G.

$= \sup_{k\in K} A(xk)$, for all x$\in$G .

$= \overline{A}(xK)$.

$\overline{A}((xK)^{-1}) = \overline{A}(xK)$.

Hence, $\overline{A}$ is a multi-fuzzy subgroup of G/K .

B. Definition 3.2

Let A be a normal multi-fuzzy subgroup of a group G with the identity element 'e'. Let K = { x$\in$G / A(x) = A(e) }. Consider $\overline{A}$ = ($\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots, \overline{A}_k$) which is defined by $\overline{A}(xK) = \sup_{k\in K} A(xk)$, for all x$\in$G where each $\overline{A}_i : G/K \to [0,1]$. Then the multi-fuzzy subgroup $\overline{A}$ of G/K is called a multi-fuzzy quotient group or quotient multi-fuzzy subgroup of A by K.

C. Remarks3.3

1) $\overline{A}$ is not a normal multi-fuzzy quotient group of G/K , Since, $\overline{A}(xKyK) \neq \overline{A}(yKxK)$.

2) Consider $\overline{A}$ = ($\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots, \overline{A}_k$) which is defined by $\overline{A}(xK) = A(x)$, for all x$\in$G, where each $\overline{A}_i : G/K \to [0,1]$. Then $\overline{A}$ is a normal multi-fuzzy quotient group of G/K .

D. Theorem3.4

$\overline{A} = (\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots, \overline{A}_k)$ is a multi-fuzzy quotient group of a group G/K iff each $\overline{A}_i$, i = 1,2,…k, is a fuzzy quotient group of a group G/K .

1) *Proof:* Let $\overline{A} = (\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots, \overline{A}_k)$ be a multi-fuzzy quotient group of a group G/K . Then,

$\Leftrightarrow \overline{A}(xyK) \geq \min\{ \overline{A}(xK) , \overline{A}(yK)\}$ and $\overline{A}(x^{-1}K) = \overline{A}(xK)$.

$\Leftrightarrow \overline{A}_i(xyK) \geq \min\{ \overline{A}_i(xK) , \overline{A}_i(yK)\}$ and $\overline{A}_i(x^{-1}K) = \overline{A}_i(xK)$, for all i = 1, 2, …, k.

$\Leftrightarrow \overline{A}_i$, i = 1,2,…,k , is a fuzzy quotient group of a group G/K .

E. Remark3.5

If $\overline{A} = (\overline{A}_1, \overline{A}_2, \overline{A}_3, \ldots, \overline{A}_k)$ is not a multi-fuzzy quotient group of a group G/K, then there is atleast one $\overline{A}_i$, i=1,2,…,k , is not a fuzzy quotient group of a group G/K.

F. Theorem 3.6

If $\overline{A}$ is a multi-fuzzy quotient group of a group G/K, then $\overline{A}(xK) \leq \overline{A}(eK)$, for all $x \in G$, where e is the identity element of G.

1) Proof: Let the element $x \in G$, where e is the identity element of G.

$$
\begin{aligned}
\text{Now,}\quad \overline{A}(eK) &= \overline{A}(xx^{-1}K) \\
&\geq \min\{\overline{A}(xK), \overline{A}(x^{-1}K)\} \\
&= \overline{A}(xK).
\end{aligned}
$$

Therefore, $\overline{A}(eK) \geq \overline{A}(xK)$, for all $x \in G$.

G. Theorem3.7

$\overline{A}$ is a multi-fuzzy quotient group of a group G/K if and only if

$\overline{A}(xKy^{-1}K) \geq \min\{\overline{A}(xK), \overline{A}(yK)\}$, for all x and y in G .

1) Proof: Assume that $\overline{A}$ is a multi-fuzzy quotient group of a group G/K.

$$
\begin{aligned}
\text{We have,}\quad \overline{A}(xKy^{-1}K) &\geq \min\{\overline{A}(xK), \overline{A}(y^{-1}K)\} \\
&\geq \min\{\overline{A}(xK), \overline{A}(yK)\}
\end{aligned}
$$

Therefore, $\overline{A}(xKy^{-1}K) \geq \min\{\overline{A}(xK), \overline{A}(yK)\}$, for all x and y in G.

Conversely, if $\overline{A}(xKy^{-1}K) \geq \min\{\overline{A}(xK), \overline{A}(yK)\}$, then

$$
\begin{aligned}
\overline{A}(x^{-1}K) &= \overline{A}(ex^{-1}K) \\
&\geq \min\{\overline{A}(eK), \overline{A}(xK)\} \\
&= \overline{A}(xK).
\end{aligned}
$$

Therefore, $\overline{A}(x^{-1}K) \geq \overline{A}(xK)$, for all x in G.

Hence, $\overline{A}((x^{-1})^{-1}K) \geq \overline{A}(x^{-1}K)$ and $\overline{A}(xK) \geq \overline{A}(x^{-1}K)$.

herefore, $\overline{A}(x^{-1}K) = \overline{A}(xK)$, for all x in G.

Now, replace y by y^{-1}, then

$$
\begin{aligned}
\overline{A}(xyK) &= \overline{A}(x(y^{-1})^{-1}K) \\
&\geq \min\{\overline{A}(xK), \overline{A}(y^{-1}K)\} \\
\overline{A}(xyK) &\geq \min\{\overline{A}(xK), \overline{A}(y^{-1}K)\}, \text{ for all x and y in G.}
\end{aligned}
$$

Hence, $\overline{A}$ is a multi-fuzzy quotient group of a group G/K.

H. Theorem 3.8

If $\overline{A}$ and $\overline{B}$ are two multi-fuzzy quotient groups of a group G/K, then $\overline{A} \cap \overline{B}$ is a multi-fuzzy quotient group of G/K.

1) Proof: It is trivial.

I. Remark 3.9

The intersection of a family of multi-fuzzy quotient groups of a group G/K, is a multi-fuzzy quotient group of a group G/K.

IV. PROPERTIES OF MULTI-FUZZY QUOTIENT GROUP $\overline{A}$ DETERMINED BY A AND K UNDER HOMOMORPHISM AND ANTI-HOMOMORPHISM

In this section, we discuss some of the properties of multi-fuzzy quotient group of a group G/K determined by A and K under homomorphism and anti-homomorphism.

A. *Theorem 4.1*

Let G and G′ be any two groups. Let f: G → G′ be a homomorphism and onto. Let $\overline{A}$ be a multi-fuzzy quotient group of G/K. Then f($\overline{A}$) is a multi-fuzzy quotient group of G'/K', if $\overline{A}$ has sup property and $\overline{A}$ is f-invariant and $f(\overline{A}) = \overline{f(A)}$.

1) Proof: Let $\overline{A}$ be a multi-fuzzy quotient group of G/K.

$$
\begin{aligned}
f(\overline{A})(f(x)f(y)K) &= (f(\overline{A}))(f(xy)K) \\
&= \overline{A}(xyK) \\
&\geq \min\{\overline{A}(xK), \overline{A}(yK)\} \\
&= \min\{(f(\overline{A}))(f(x)K), (f(\overline{A}))(f(y)K)\} \\
f(\overline{A})(f(x)f(y)K) &\geq \min\{(f(\overline{A}))(f(x)K), (f(\overline{A}))(f(y)K)\}. \\
f(\overline{A})([f(x)]^{-1}K) &= f(\overline{A})[f(x^{-1})K] \\
&= \overline{A}(x^{-1}K) \\
&= \overline{A}(xK) \\
&= f(\overline{A})[f(x)K] \\
f(\overline{A})([f(x)]^{-1}K) &= f(\overline{A})[f(x)K].
\end{aligned}
$$

Hence, f ($\overline{A}$) is amulti-fuzzy quotient group of G'/K'.

$$
\begin{aligned}
\text{Also, } \overline{f(A)}(yK) &= \sup_{k\in K} f(A)(yk), \text{ for all } y\in G'. \\
&= \sup_{k\in K} f(A)(f(x)k), \text{ f is onto and } x\in G. \\
&= \sup_{k\in K} A(xk), \text{ for all } x\in G \\
&= \overline{A}(xK) \\
&= f(\overline{A})(f(x)K) \\
&= f(\overline{A})(yK). \\
\text{Hence, } \overline{f(A)}(yK) &= f(\overline{A})(yK).
\end{aligned}
$$

B. *Theorem 4.2*

Let G and G′ be any two groups. Let f: G → G′ be a homomorphism. Let $\overline{B}$ be a multi-fuzzy quotient group of G'/K'. Then $f^{-1}(\overline{B})$ is a multi-fuzzy quotient group of G/K and $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

1) Proof: Let $\overline{B}$ be a multi-fuzzy quotient group of G'/K'. For all x, y in G,

$$
f^{-1}(\overline{B})(xyK) = \overline{B}(f(xy)K)
$$

$$= \overline{B}\,(f(x)f(y)K)$$
$$\geq \min\{\overline{B}\,(f(x)K)\,,\ \overline{B}\,(f(y)K)\}$$
$$\geq \min\{\,f^{-1}(\overline{B}\,)(xK)\,,\ f^{-1}(\overline{B}\,)(yK)\}.$$

That is, $f^{-1}(\overline{B}\,)(xyK) \geq \min\{\,f^{-1}(\overline{B}\,)(xK)\,,\ f^{-1}(\overline{B}\,)(yK)\}$.

$$f^{-1}(\overline{B}\,)(x^{-1}K) = \overline{B}\,(f(x^{-1})K)$$
$$= \overline{B}\,((f(x))^{-1}K)$$
$$= \overline{B}\,(f(x)K)$$
$$= f^{-1}(\overline{B}\,)(xK).$$

That is, $f^{-1}(\overline{B}\,)(x^{-1}K) = f^{-1}(\overline{B}\,)(xK)$.

Hence, $f^{-1}(\overline{B}\,)$ is a multi-fuzzy quotient group of G/K.

Also, $\overline{f^{-1}(B)}\,(xK) = \sup_{k\in K} f^{-1}(B)(xk)$, for all $x\in G$.

$$= \sup_{k\in K} B(f(x)k)\text{, for all } x\in G.$$
$$= \overline{B}\,(f(x)K)$$
$$= f^{-1}(\overline{B}\,)(xK).$$

Hence, $\overline{f^{-1}(B)}\,(xK) = f^{-1}(\overline{B}\,)(xK)$.

C. Theorem 4.3

Let G and G′ be any two groups. Let f: G → G′be an anti-homomorphism and onto. Let $\overline{A}$ be a multi-fuzzy quotient group of G/K. Then f($\overline{A}$) is a multi-fuzzy quotient group of G'/K', if $\overline{A}$ has sup property and $\overline{A}$ is f-invariant and $f(\overline{A}) = \overline{f(A)}$.

1) Proof: Let $\overline{A}$ be a multi-fuzzy quotient group of G/K.

if $(\overline{A}\,)(f(x)f(y)K) = (f\,(\overline{A}\,))(f(yx)K)$
$$= \overline{A}\,(yxK)$$
$$\geq \min\{\overline{A}\,(yK)\,,\overline{A}\,(xK)\,\}$$
$$\geq \min\{\overline{A}\,(xK)\,,\overline{A}\,(yK)\,\}$$
$$= \min\{\,(f\,(\overline{A}\,))(f(x)K)\,,\ (f\,(\overline{A}\,)(f(y)K)\,\}$$

$$f\,(\overline{A}\,)(f(x)f(y)K) \geq \min\{\,(f\,(\overline{A}\,))(f(x)K)\,,\ (f\,(\overline{A}\,)(f(y)K)\,\}.$$

$$f\,(\overline{A}\,)([f(x)]^{-1}K) = f\,(\overline{A}\,)[f(x^{-1})K]$$
$$= \overline{A}\,(x^{-1}K)$$
$$= \overline{A}\,(xK)$$
$$= f\,(\overline{A}\,)[f(x)K].$$

$$f\,(\overline{A}\,)([f(x)]^{-1}\,K) = f\,(\overline{A}\,)[f(x)K].$$

Hence, f ($\overline{A}$) is amulti-fuzzy quotient group of G'/K'.

Also, $\overline{f(A)}\,(yK) = \sup_{k\in K} f(A)(yk)$, for all $y\in G'$.

$= \sup_{k\in K} f(A)(f(x)k)$, f is onto and $x\in G$.

$= \sup_{k\in K} A(xk)$, for all $x\in G$.

$= \overline{A}(xK)$

$= f(\overline{A})(f(x)K)$

$= f(\overline{A})(yK)$.

Hence, $\overline{f(A)}(yK) = f(\overline{A})(yK)$.

D. Theorem 4.4

Let G and G′ be any two groups. Let f: G → G′ be an anti-homomorphism. Let $\overline{B}$ be a multi-fuzzy quotient group of G'/K'.

Then $f^{-1}(\overline{B})$ is a multi-fuzzy quotient group of G/K and $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Let $\overline{B}$ be a multi-fuzzy quotient group of G'/K'.

$f^{-1}(\overline{B})(xyK) = \overline{B}(f(xy)K)$

$= \overline{B}(f(y)f(x)K)$

$\geq \min\{\overline{B}(f(y)K), \overline{B}(f(x)K)\}$

$\geq \min\{\overline{B}(f(x)K), \overline{B}(f(y)K)\}$

$\geq \min\{f^{-1}(\overline{B})(xK), f^{-1}(\overline{B})(yK)\}$.

$f^{-1}(\overline{B})(xyK) \geq \min\{f^{-1}(\overline{B})(xK), f^{-1}(\overline{B})(yK)\}$.

$f^{-1}(\overline{B})(x^{-1}K) = \overline{B}(f(x^{-1})K)$

$= \overline{B}((f(x))^{-1}K)$

$= \overline{B}(f(x)K)$

$= f^{-1}(\overline{B})(xK)$.

$f^{-1}(\overline{B})(x^{-1}K) = f^{-1}(\overline{B})(xK)$.

Hence, $f^{-1}(\overline{B})$ is a multi-fuzzy quotient subgroup of G/K.

Also, $\overline{f^{-1}(B)}(xK) = f^{-1}(B)(xk)$, for all $x\in G$.

$= \sup_{k\in K} B(f(x)k)$, for all $x\in G$.

$= \overline{B}(f(x)K)$

$= f^{-1}(\overline{B})(xK)$.

Hence, $\overline{f^{-1}(B)}(xK) = f^{-1}(\overline{B})(xK)$.

V. CONCLUSION

In this paper we define a new algebraic structure of multi-fuzzy quotient group of a group and discussed some of its related properties under homomorphism and anti-homomorphism. Theseare very helpful to the development of the theory of multi-fuzzy set. The purpose of this study is to implement the fuzzy set theory and the group theory in multi-fuzzy set. Characterization of multi-fuzzy quotient group of a group are given.

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