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Common Fixed-Point Theorems in Complex Valued Metric Spaces Using New Contractive Condition

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Abstract: In this article, we introduce some notions on complex valued metric spaces. Also, we obtain common fixed point for a pair of multivalued mappings satisfying set inclusion type contractive condition. Moreover an example reveals that we can obtain common fixed point under this new contractive condition.

Keywords: Common fixed point, Complete metric space, Multivalued map-pings, Complex valued metric spaces.

Mathemetics Subject Classification: 54H25, 47H10.

I. INTRODUCTION AND PRELIMINARIES

For the last three decades, the numerous articles are being published regarding Fixed point theory by the great impact on the most powerful theorem “Banach Contraction Mapping Principle". In 1969, S.B. Nadler [1] introduced the notion of Libschitz inequality for multivalued mappings and generalized the classical Banach contraction theorem. Recently, Azam et al. [2] established a remarkable result on generalizing the Banach Contraction Mapping Principle. They defined a metric space, called complex valued metric space where the set of complex numbers is used instead of the set of real numbers. They also obtained some generalizations of Banach's result in the newly generalized space. Furthermore they overcome the rational expressions in cone metric spaces with the complex valued metric spaces. The Fixed point theory in complex valued metric spaces is very useful not only in Mathematical Analysis but also in Physics, Chemistry, Engineering, etc,. Many generalizations of Theorem 4 of [2] were established in directions including replacing the constants with functions or generalizing the contractive condition, etc,. See [3] - [10].

Ahmed et al. [11] introduced the notion of multivalued mapping in complex valued metric space and established some common fixed point theorems for multivalued mappings under new contractive condition which is in form 'element-set relation'. In the sequel, research articles have been published on generalizing the results of [11]. See [12] - [13].In this paper, we introduce some new notions on complex valued metric spaces. Also, we generalize the results of Azam et al. [2] and obtain some new fixed point theorems in complex valued metric spaces. An example is given to illustrate our result.

The following are the prerequisite for the sake of understanding.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

It follows that $z_1 \precsim z_2$ if one of the following condition is satisfied:

$$e(z_1) = Re(z_2),\ \ Im(z_1) < Im(z_2),$$

$$(\quad) < Re(z_2),\ \ Im(z_1) = Im(z_2),$$

$$e(z_1) < Re(z_2),\ \ Im(z_1) < Im(z_2),$$

$$(z_1) = Re(z_2),\ \ Im(z_1) = Im(z_2),$$

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|,$$

$$z_1 \precsim z_2, z_2 \prec z_3 \Longrightarrow z_1 \prec z_3 .$$

A. Definition

Let X be a nonempty set. Then (X, d) is called a complex valued metric space with metric d if the mapping $d: X \times X \rightarrow \mathbb{C}$, satisfies the following conditions:

1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

2) $d(x,y) = d(y,x)$, for all $x, y \in X$.
3) $d(x,y) \precsim d(x,z) + d(z,y)$, for all $x, y, z \in X$.

B. Definition

Let (X,d) be a complex valued metric space. Then
a sequence (x_n) in X is a Cauchy sequence if for every $0 \prec c \in \mathbb{C}$, there exists an integer N such that $d(x_n, x_m) \prec c$ for all $n, m \geq N$.a sequence (x_n) in X converges to an element $x \in X$ if for every $0 \prec c \in \mathbb{C}$,
there exists an integer N such that $d(x_n, x) \prec c$ for all $n \geq N$.

1) *Lemma 1.1:* Let (X,d) be a complex valued metric space and let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.
2) *Lemma 1.2: :* Let (X,d) complex valued metric space and let (x_n) be a sequence in X. Then (x_n) is a Cauchy sequence if and only if $|d(x_n, x_m)| \to 0$ as $m \geq n \to \infty$. Let (X,d) be a complex valued metric space. Denote the family of nonempty, closed and bounded subsets of complex valued metric space (X,d) by $CB(X)$.
3) *Definition 1.3:* [11] Let (X,d) be a complex valued metric space. Let $T: X \to CB(X)$ be a multivalued map. For $x \in X$ and $A \in CB(X)$, define
$$W_x(A) = \{d(x,a): a \in A\}.$$
Thus, for $x, y \in X$, $Ty \in CB(X)$
$W_x(Ty) = \{d(x,u): u \in Ty\}$.Here we give the corrected version of Definition 5 of [11].
4) *Definition 1.4:* [11] Let (X,d) be a complex valued metric space. A subset A of X is called bounded from below if there exists some $z \in \mathbb{C}$ such that $z \precsim d(a,b)$ for all $a, b \in A$.

5) *Definition 1.5:* [11] Let (X,d) be a complex valued metric space. A multivalued map $F: X \to 2^{\mathbb{C}}$ is called bounded from below if for each $x \in X$, there exists $z_x \in \mathbb{C}$ such that $z_x \precsim u$ for all $u \in Fx$.
6) *Definition 1.6:* [11] Let (X,d) be a complex valued metric space. A multivalued map $T: X \to CB(X)$ is said to have the lower bound property (*l.b* property) on X, if for any $x \in X$, the multivalued map $F: X \to 2^{\mathbb{C}}$ defined by
$$Fx = W_x(Ty)$$

is bounded from below. That is, for $x, y \in X$, there exists an element $I_x(Ty) \in \mathbb{C}$ such that $I_x(Ty) \precsim u$, for all $u \in W_x(Ty)$, where $I_x(Ty)$ is called a lower bound of T associated with (x,y).

7) *Definition 1.7*: [11] Let (X,d) be a complex valued metric space. A multivalued valued map $T: X \to CB(X)$ is said to have the greatest lower bound property *(g.l.b property)* on X, if a greatest lower bound of $W_x(Ty)$ exists in $\mathbb{C}$ for all $x, y \in X$. Denote $d(x, Ty)$ by the *g.l.b* of $W_x(Ty)$. That is,
$$d(x, Ty) = \inf\{d(x,u): u \in Ty\}.$$

Now let us introduce some notions on the set of complex numbers as follows. Denote
$$u(z_0) = \{z \in \mathbb{C}: z_0 \precsim z\} \quad \text{and}$$
$$u(a,B) = \bigcap_{b \in B} u(d(a,b)) = \bigcap_{b \in B} (z \in \mathbb{C}: 0 \precsim d(a,b) \precsim z)$$
$$u(A,B) = (\bigcup_{a \in A} u(a,B)) \bigcap (\bigcup_{b \in B} u(b,A)), A, B \in CB(X).$$

8) *Definition 1.8:* Let (X,d) be a complex valued metric space. A nonempty subset A of X is called bounded from above if there exists some $z^* \in \mathbb{C}$ such that $d(a,b) \precsim z^*$ for all $a, b \in A$.
9) *Definition 1.9:* Let (X,d) be a complex valued metric space. A nonempty subset A of X is called bounded if there exist z and z^* in $\mathbb{C}$ such that $z \precsim d(a,b) \precsim z^*$ for all $a, b \in A$.
10) *Lemma 1.3:* Let $\mathbb{C}$ be the set of complex numbers with the partial order $\precsim$ defined on $\mathbb{C}$. A nonempty subset A in $\mathbb{C}$ is bounded with an upper bound $z^* = x^* + iy^*$ and a lower bound $z = x + iy$ iff A can be covered by the rectangle with vertices $(x,y), (x^*,y), (x,y^*)$ and (x^*,y^*).
11) *Definition 1.10:* Let (X,d) be a complex valued metric space. A multivalued map $F: X \to 2^{\mathbb{C}}$ is called bounded from above if for each $x \in X$, there exists $z_x^* \in \mathbb{C}$ such that $u \precsim z_x^*$ for all $u \in Fx$.
12) *Definition 1.11:* [13] Let (X,d) be a complex valued metric space. Let $S: X \to CB(X)$ be a multivalued mapping and $\alpha: X \to [0,1)$ such that $\alpha(u) \leq \alpha(x)$ for all $u \in Sx$ and $\forall x \in X$.
Let Ω be a family of functions

$\Omega = \{\alpha: \alpha: X \to [0,1)$ such that $\alpha(u) \le \alpha(x)$ for all $u \in Sx$ and $\forall\, x \in X\}$.

II. MAIN RESULTS

In this section, we establish some fixed point theorems which generalize recent results including Theorem 4 of [2].
Let ψ be a family of non-decreasing functions $\phi: \mathbb{C} \to \mathbb{C}$ such that $\phi(0) = 0$ and $\phi(t) \precsim t$ when $t \neq 0$.

A. *Theorem 2.1*

Let (X, d) be a complete complex valued metric space and let the functions $\alpha, \beta, \gamma, \delta, \eta, \mu, \xi$ be in Ω such that $\alpha + \beta + \gamma + 2\delta + 2\eta + \mu < 1$ and $\phi \in \psi$. If $S, T: X \to CB(X)$ be multivalued mappings with *g.l.b* property such that

$$u(\phi(\alpha(x)d(x,y) + \beta(x)d(x,Sx) + \gamma(x)d(y,Ty) + \delta(x)\frac{d(x,Sx)d(x,Ty)}{1+d(x,y)}$$
$$+ \eta(x)\frac{d(y,Sx)d(y,Ty)}{1+d(x,y)} + \mu(x)\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \xi(x)\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)}))$$
$$\subseteq u(Sx, Ty) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

for all $x, y \in X$. Then S and T have a common fixed point.

1) *Proof:* Let x_0 be an arbitrary point in X and $x_1 \in Sx_0$. For $x = x_0$ and $y = x_1$, our hypothesis (1) implies that

$$u(\phi(\alpha(x_0)d(x_0,x_1) + \beta(x_0)d(x_0,Sx_0) + \gamma(x_0)d(x_1,Tx_1) + \delta(x_0)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}$$
$$+ \eta(x_0)\frac{d(x_1,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)} + \mu(x_0)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)} + \xi(x_0)\frac{d(x_1,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}))$$
$$\subseteq u(Sx_0, Tx_1)$$

Since $d(x_1, Sx_0) = 0$, we get

$$\phi(\alpha(x_0)d(x_0,x_1) + \beta(x_0)d(x_0,Sx_0) + \gamma(x_0)d(x_1,Tx_1) + \delta(x_0)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}$$
$$+\mu(x_0)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}) \in u(Sx_0, Tx_1)$$

Then there exists $x_2 \in Tx_1$ such that $z \in u(Sx_0, Tx_1)$ implies $z \in u(x_2, Sx_0)$ and for all $s \in Sx_0$, $z \in u(x_2, Sx_0) \Rightarrow z \in u\big(d(x_2, s)\big), x_2 \in Tx_1$. So $z \in u(Sx_0, Tx_1)$ implies $z \in u(d(x_2, x_1))$, some $x_2 \in Tx_1$ and $x_1 \in Sx_0$. It follows that

$$\phi(\alpha(x_0)d(x_0,x_1) + \beta(x_0)d(x_0,Sx_0) + \gamma(x_0)d(x_1,Tx_1) + \delta(x_0)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}$$
$$+\mu(x_0)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}) \in u(d(x_2, x_1))$$

By the definition of u, we have that

$$d(x_2,x_1) \precsim \phi(\alpha(x_0)d(x_0,x_1) + \beta(x_0)d(x_0,Sx_0) + \gamma(x_0)d(x_1,Tx_1)$$
$$+\delta(x_0)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)} + \mu(x_0)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)})$$
$$\Rightarrow \quad d(x_2,x_1) \precsim \alpha(x_0)d(x_0,x_1) + \beta(x_0)d(x_0,Sx_0) + \gamma(x_0)d(x_1,Tx_1)$$
$$+\delta(x_0)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)} + \mu(x_0)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}$$

Since $d(x_0, Sx_1) \precsim d(x_0, x_1)$ and $d(x_0, Tx_1) \precsim d(x_0, x_2)$, we have

$$d(x_2,x_1) \precsim \alpha(x_0)d(x_0,x_1) + \beta(x_0)d(x_0,x_1) + \gamma(x_0)d(x_1,x_2)$$
$$+\delta(x_0)\frac{d(x_0,x_1)d(x_0,x_2)}{1+d(x_0,x_1)} + \mu(x_0)\frac{d(x_0,x_1)d(x_1,x_2)}{1+d(x_0,x_1)}$$

Now, we get that

$$|d(x_2,x_1)| \le |\alpha(x_0)||d(x_0,x_1)| + |\beta(x_0)||d(x_0,x_1)| + |\gamma(x_0)||d(x_1,x_2)|$$
$$+|\delta(x_0)||\frac{d(x_0,x_1)d(x_0,x_2)}{1+d(x_0,x_1)}| + |\mu(x_0)||\frac{d(x_0,x_1)d(x_1,x_2)}{1+d(x_0,x_1)}|$$
$$\le |\alpha(x_0)||d(x_0,x_1)| + |\beta(x_0)||d(x_0,x_1)| + |\gamma(x_0)||d(x_1,x_2)|$$
$$+|\delta(x_0)||d(x_0,x_2)| + |\mu(x_0)||d(x_1,x_2)|$$
$$= \big(\alpha(x_0) + \beta(x_0) + \delta(x_0)\big)|d(x_0,x_1)| + (\gamma(x_0) + \delta(x_0) + +\mu(x_0))$$
$$|d(x_1,x_2)|$$

$\Longrightarrow |d(x_1,x_2)| \leq \frac{\alpha(x_0)+\beta(x_0)+\delta(x_0)}{1-(\gamma(x_0)+\delta(x_0)++\mu(x_0))}|d(x_0,x_1)|$,

i.e., $|d(x_1,x_2)| \leq \lambda_1 |d(x_0,x_1)|$, ...(2)

where $\lambda_1 = \frac{\alpha(x_0)+\beta(x_0)+\delta(x_0)}{1-(\gamma(x_0)+\delta(x_0)++\mu(x_0))}$.

Put $x = x_2$ and $y = x_1$ in (1), we have

$$u(\phi(\alpha(x_2)d(x_2,x_1) + \beta(x_2)d(x_2,Sx_2) + \gamma(x_2)d(x_1,Tx_1)$$
$$+\delta(x_2)\frac{d(x_2,Sx_2)d(x_2,Tx_1)}{1+d(x_2,x_1)} + \eta(x_2)\frac{d(x_1,Sx_2)d(x_1,Tx_1)}{1+d(x_2,x_1)}$$
$$+\mu(x_2)\frac{d(x_2,Sx_2)d(x_1,Tx_1)}{1+d(x_2,x_1)} + \xi(x_2)\frac{d(x_1,Sx_2)d(x_2,Tx_1)}{1+d(x_2,x_1)}))$$
$$\subseteq u(Sx_2,Tx_1)$$

Now $x_2 \in Tx_1$ implies $d(x_2,Tx_1) = 0$, so

$$u(\phi(\alpha(x_2)d(x_2,x_1) + \beta(x_2)d(x_2,Sx_2) + \gamma(x_2)d(x_1,Tx_1)$$
$$+ \eta(x_2)\frac{d(x_1,Sx_2)d(x_1,Tx_1)}{1+d(x_2,x_1)} + \mu(x_2)\frac{d(x_2,Sx_2)d(x_1,Tx_1)}{1+d(x_2,x_1)})$$
$$\subseteq u(Sx_2,Tx_1)$$

Since there exists x_3 in Sx_2 such that $z \in u(Sx_2,Tx_1) \Longrightarrow z \in u(x_3,Tx_2)$. It follows that $z \in u(Sx_2,Tx_1) \Longrightarrow z \in u(d(x_3,t))$ for all $t \in Tx_1$. Therefore $z \in u(Sx_2,Tx_1) \Longrightarrow z \in u(d(x_3,x_2))$. Now we have

$$\phi(\alpha(x_2)d(x_2,x_1) + \beta(x_2)d(x_2,Sx_2) + \gamma(x_2)d(x_1,Tx_1)$$
$$+ \eta(x_2)\frac{d(x_1,Sx_2)d(x_1,Tx_1)}{1+d(x_2,x_1)} + \mu(x_2)\frac{d(x_2,Sx_2)d(x_1,Tx_1)}{1+d(x_2,x_1)}$$
$$\in u(d(x_3,x_2))$$

By the definition of u and g.l.b property, we have

$$d(x_3,x_2) \precsim \phi(\alpha(x_2)d(x_2,x_1) + \beta(x_2)d(x_2,x_3) + \gamma(x_2)d(x_1,x_2)$$
$$+ \eta(x_2)\frac{d(x_1,x_3)d(x_1,x_2)}{1+d(x_2,x_1)} + \mu(x_2)\frac{d(x_2,x_3)d(x_1,x_2)}{1+d(x_2,x_1)}$$
$$\precsim \alpha(x_2)d(x_2,x_1) + \beta(x_2)d(x_2,x_3) + \gamma(x_2)d(x_1,x_2)$$
$$+ \eta(x_2)\frac{d(x_1,x_3)d(x_1,x_2)}{1+d(x_2,x_1)} + \mu(x_2)\frac{d(x_2,x_3)d(x_1,x_2)}{1+d(x_2,x_1)}$$

Because $d(x_2,x_1) \precsim 1 + d(x_2,x_1)$, we obtain

$$d(x_3,x_2) \precsim \alpha(x_2)d(x_2,x_1) + \beta(x_2)d(x_2,x_3) + \gamma(x_2)d(x_1,x_2)$$
$$+ \eta(x_2)d(x_1,x_3) + \mu(x_2)d(x_3,x_2)$$

By the property of functions in Ω,

$$d(x_3,x_2) \precsim \alpha(x_0)d(x_2,x_1) + \beta(x_0)d(x_2,x_3) + \gamma(x_0)d(x_1,x_2)$$
$$+ \eta(x_0)d(x_1,x_3) + \mu(x_0)d(x_3,x_2)$$

Now , by taking absolute value of complex numbers, we get

$$|d(x_3,x_2)| \leq |\alpha(x_0)||d(x_2,x_1)| + |\beta(x_0)||d(x_2,x_3)| + |\gamma(x_0)||d(x_1,x_2)|$$
$$+ |\eta(x_0)||d(x_1,x_3)| + |\mu(x_0)||d(x_3,x_2)|$$
$$\Longrightarrow \quad \leq \left(\alpha(x_0) + \gamma(x_0) + \eta(x_0)\right)|d(x_2,x_1)|$$
$$+(\beta(x_0) + \eta(x_0) + \mu(x_0))|d(x_3,x_2)|$$

Therefore

$$|d(x_3,x_2)| \leq \frac{\alpha(x_0) + \gamma(x_0) + \eta(x_0)}{1 - (\beta(x_0) + \eta(x_0) + \mu(x_0))}\ |d(x_2,x_1)|$$

i.e., $|d(x_3,x_2)| \leq \lambda_2\ |d(x_2,x_1)|$, ...(3)

where $\lambda_2 = \frac{\alpha(x_0)+\gamma(x_0)+\eta(x_0)}{1-(\beta(x_0)+\eta(x_0)+\mu(x_0))}$.

Let $\lambda = \max\{\lambda_1,\lambda_2\}$.

Thus from (2) and (3), there exists a sequence (x_n) in X such that $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$, $n = 0,1,2,3,\cdots$ satisfying the criteria

$$|d(x_{n+1}, x_n)| \leq \lambda^n |d(x_1, x_0)|, \text{ for all } n.$$

Now for m$\geq n$, we have

$$\begin{aligned} |d(x_n, x_m)| &\leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + \cdots + |d(x_{m-1}, x_m)| \\ &\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})|d(x_1, x_0)| \\ &\leq [\frac{\lambda^n}{1-\lambda}]|d(x_1, x_0)| \end{aligned}$$

As $m \geq n \to \infty \implies |d(x_n, x_m)| \to 0$. Therefore the sequence (x_n) is a Cauchy sequence. Since X is a complete complex valued metric space, there exists an element p in X such that (x_n) converges to p.

Now we claim that $p \in Tp$. For that, let $x = x_{2n}$, $y = p$ in (1), we have

$$\begin{aligned} &u(\phi(\alpha(x_{2n})d(x_{2n}, p) + \beta(x_{2n})d(x_{2n}, Sx_{2n}) + \gamma(x_{2n})d(p, Tp) \\ &+\delta(x_{2n})\frac{d(x_{2n}, Sx_{2n})d(x_{2n}, Tp)}{1+d(x_{2n}, p)} + \eta(x_{2n})\frac{d(p, Sx_{2n})d(p, Tp)}{1+d(x_{2n}, p)} \\ &+\mu(x)\frac{d(x_{2n}, Sx_{2n})d(p, Tp)}{1+d(x_{2n}, p)} + \xi(x_{2n})\frac{d(p, Sx_{2n})d(x_{2n}, Tp)}{1+d(x_{2n}, p)})) \\ &\subseteq u(Sx_{2n}, Tp) \end{aligned}$$

Now there exists $p_n \in Tp$ such that $z \in u(Sx_{2n}, Tp) \implies u(d(p_n, s))$, for all $s \in Sx_{2n}$. It follows that $z \in u(Sx_{2n}, Tp) \implies z \in u(d(p_n, x_{2n+1}))$. Therefore,

$$\begin{aligned} &u(\phi(\alpha(x_{2n})d(x_{2n}, p) + \beta(x_{2n})d(x_{2n}, Sx_{2n}) + \gamma(x_{2n})d(p, Tp) \\ &+\delta(x_{2n})\frac{d(x_{2n}, Sx_{2n})d(x_{2n}, Tp)}{1+d(x_{2n}, p)} + \eta(x_{2n})\frac{d(p, Sx_{2n})d(p, Tp)}{1+d(x_{2n}, p)} \\ &+\mu(x)\frac{d(x_{2n}, Sx_{2n})d(p, Tp)}{1+d(x_{2n}, p)} + \xi(x_{2n})\frac{d(p, Sx_{2n})d(x_{2n}, Tp)}{1+d(x_{2n}, p)})) \\ &\subseteq u(d(x_{2n+1}, p_n)) \end{aligned}$$

This implies that

$$\begin{aligned} d(x_{2n+1}, p_n) &\precsim \phi(\alpha(x_{2n})d(x_{2n}, p) + \beta(x_{2n})d(x_{2n}, Sx_{2n}) + \gamma(x_{2n})d(p, Tp) \\ &+\delta(x_{2n})\frac{d(x_{2n}, Sx_{2n})d(x_{2n}, Tp)}{1+d(x_{2n}, p)} + \eta(x_{2n})\frac{d(p, Sx_{2n})d(p, Tp)}{1+d(x_{2n}, p)} \\ &+\mu(x_{2n})\frac{d(x_{2n}, Sx_{2n})d(p, Tp)}{1+d(x_{2n}, p)} + \xi(x_{2n})\frac{d(p, Sx_{2n})d(x_{2n}, Tp)}{1+d(x_{2n}, p)})) \\ &\precsim \alpha(x_{2n})d(x_{2n}, p) + \beta(x_{2n})d(x_{2n}, Sx_{2n}) + \gamma(x_{2n})d(p, Tp) \\ &+\delta(x_{2n})\frac{d(x_{2n}, Sx_{2n})d(x_{2n}, Tp)}{1+d(x_{2n}, p)} + \eta(x_{2n})\frac{d(p, Sx_{2n})d(p, Tp)}{1+d(x_{2n}, p)} \\ &+\mu(x_{2n})\frac{d(x_{2n}, Sx_{2n})d(p, Tp)}{1+d(x_{2n}, p)} + \xi(x_{2n})\frac{d(p, Sx_{2n})d(x_{2n}, Tp)}{1+d(x_{2n}, p)}) \\ &\precsim \alpha(x_{2n})d(x_{2n}, p) + \beta(x_{2n})d(x_{2n}, x_{2n+1}) + \gamma(x_{2n})d(p, p_n) \\ &+\delta(x_{2n})\frac{d(x_{2n}, x_{2n+1})d(x_{2n}, p_n)}{1+d(x_{2n}, p)} + \eta(x_{2n})\frac{d(p, x_{2n+1})d(p, p_n)}{1+d(x_{2n}, p)} \\ &+\mu(x_{2n})\frac{d(x_{2n}, x_{2n+1})d(p, p_n)}{1+d(x_{2n}, p)} + \xi(x_{2n})\frac{d(p, x_{2n+1})d(x_{2n}, p_n)}{1+d(x_{2n}, p)}) \end{aligned}$$

Therefore,

$$\begin{aligned} |d(x_{2n+1}, p_n)| &\leq |\alpha(x_{2n})||d(x_{2n}, p)| + |\beta(x_{2n})||d(x_{2n}, x_{2n+1})| + |\gamma(x_{2n})||d(p, p_n)| \\ &+|\delta(x_{2n})||\frac{d(x_{2n}, x_{2n+1})d(x_{2n}, p_n)}{1+d(x_{2n}, p)}| + |\eta(x_{2n})||\frac{d(p, x_{2n+1})d(p, p_n)}{1+d(x_{2n}, p)}| \\ &+|\mu(x_{2n})||\frac{d(x_{2n}, x_{2n+1})d(p, p_n)}{1+d(x_{2n}, p)}| + |\xi(x_{2n})||\frac{d(p, x_{2n+1})d(x_{2n}, p_n)}{1+d(x_{2n}, p)}| \\ &\leq \alpha(x_0)\,|d(x_{2n}, p)| + \beta(x_0)\,|d(x_{2n}, x_{2n+1})| + \gamma(x_0)\,|d(p, p_n)| \\ &+\delta(x_0)\left|\frac{d(x_{2n}, x_{2n+1})d(x_{2n}, p_n)}{1+d(x_{2n}, p)}\right| + \eta(x_0)\left|\frac{d(p, x_{2n+1})d(p, p_n)}{1+d(x_{2n}, p)}\right| \\ &+\mu(x_0)\left|\frac{d(x_{2n}, x_{2n+1})d(p, p_n)}{1+d(x_{2n}, p)}\right| + \xi(x_0)\left|\frac{d(p, x_{2n+1})d(x_{2n}, p_n)}{1+d(x_{2n}, p)}\right| \end{aligned}$$

Since $d(x_n, p) \to 0$ as n tends to ∞, we have that $\lim_{n\to\infty}|d(p, p_n)| \leq \gamma(x_0)|d(p, p_n)|$.

Because $\gamma(x_0) < 1$, we get tht $|d(p, p_n)| \to 0$ as $n \to \infty$. Hence $\lim_{n\to\infty} p_n = p$.

Since $p_n \in Tp$ for all n and Tp contains all its limit points, $p \in Tp$.
Similarly we can prove that $p \in Sp$.
Thus p is a common fixed point of S and T, i.e., $p \in Sp \cap Tp$.

B. Corollary 2.1

Let (X,d) be a complete complex valued metric space and let $\alpha,\beta,\gamma,\delta,\eta,\mu,\xi \in \Omega$ such that $\alpha+\beta+\gamma+2\delta+2\eta+\mu<1$ and $k \in [0,1]$. If $S,T: X \to CB(X)$ be the multivalued mappings with *g.l.b* property such that

$$u(k(\alpha(x)d(x,y)+\beta(x)d(x,Sx)+\gamma(x)d(y,Ty)+\delta(x)\tfrac{d(x,Sx)d(x,Ty)}{1+d(x,y)} \\ +\eta(x)\tfrac{d(y,Sx)d(y,Ty)}{1+d(x,y)}+\mu(x)\tfrac{d(x,Sx)d(y,Ty)}{1+d(x,y)}+\xi(x)\tfrac{d(y,Sx)d(x,Ty)}{1+d(x,y)})) \\ \subseteq u(Sx,Ty)$$

for all $x,y \in X$. Then there exists a fixed point p of S and T such that $p \in Sp \cap Tp$.

1) Proof: Consider the function $\phi(z)=kz$, for $k \in [0,1]$. Now by Theorem 2.1, we can easily be proved.

C. Corollary 2.2

Let (X,d) be a complete complex valued metric space and let $\alpha,\beta,\gamma,\delta,\eta,\mu,\xi \in \Omega$ such that $\alpha+\beta+\gamma+2\delta+2\eta+\mu<1$. If a multivalued mapping $T: X \to CB(X)$ with *g.l.b* property such that

$$u(\alpha(x)d(x,y)+\beta(x)d(x,Tx)+\gamma(x)d(y,Ty)+\delta(x)\tfrac{d(x,Tx)d(x,Ty)}{1+d(x,y)} \\ +\eta(x)\tfrac{d(y,Tx)d(y,Ty)}{1+d(x,y)}+\mu(x)\tfrac{d(x,Tx)d(y,Ty)}{1+d(x,y)}+\xi(x)\tfrac{d(y,Tx)d(x,Ty)}{1+d(x,y)}) \\ \subseteq u(Tx,Ty)$$

for all $x,y \in X$. Then T has a fixed point p in X.

1) Proof: If we take $S=T$ in Theorem 2.1, we can obtain that $p \in Tp$, $p \in X$.

Let (X,d) be a complete complex valued metric space and let $\alpha,\beta,\gamma,\delta,\eta,\mu,\xi \in \Omega$ such that $\alpha+\beta+\gamma+2\delta+2\eta+\mu<1$. If $S,T: X \to CB(X)$ be the multivalued mappings with *g.l.b* property such that

$$u(k(\alpha(x)d(x,y)+\beta(x)d(x,Sx)+\gamma(x)d(y,Ty)+\delta(x)\tfrac{d(x,Sx)d(x,Ty)}{1+d(x,y)} \\ +\eta(x)\tfrac{d(y,Sx)d(y,Ty)}{1+d(x,y)}+\mu(x)\tfrac{d(x,Sx)d(y,Ty)}{1+d(x,y)}+\xi(x)\tfrac{d(y,Sx)d(x,Ty)}{1+d(x,y)})) \\ \subseteq u(Sx,Ty)$$

$(x)=\xi$ such that $\alpha+\beta+\gamma+2\delta+2\eta+\mu<1$. Now by the Theorem 2.1, there exists a point p in X such that $p \in Sp \cap Tp$.

D. Corollary 2.4

Let (X,d) be a complete complex valued metric space and let the self-mappings $S,T: X \to CB(X)$ satisfy

$$u(\alpha d(x,y)+\mu\,\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}) \subseteq u(Sx,Ty)$$

for all $x,y \in X$, where α and μ are non-negative real numbers provided $\alpha+\mu<1$. Then S and T have a unique common fixed point.

1) Proof: Take $\beta=\gamma=\delta=\eta=\xi=0$ in Corollary 2.3, we obtain a fixed point p of S and T. Let q be an another common fixed point of S and T. Now by our hypothesis, we have that

$$u(\alpha d(p,q)+\mu\,\frac{d(p,Sp)d(q,Tq)}{1+d(p,q)}) \subseteq u(Sp,Tq)$$

i.e., $u\big(\alpha d(p,q)\big) \subseteq u\big(d(p,q)\big)$

$\Rightarrow \quad d(p,q) \precsim \alpha d(p,q)$

$$\Rightarrow (1-\alpha)d(p,q) \precsim 0$$

$$\Rightarrow \quad p = q.$$

Hence S and T have a unique common fixed point.

a) *Remark 2.1:* Let (X,d) be a complete metric space and $\mathbb{C} = \mathbb{R}$. If $S = T$ is a self-mapping on X to itself, then $u\big(\alpha d(x,y)\big) \subseteq u(d(Tx,Ty))$ is a set version of contractive condition.

b) *Example 2.1:* Let $X = [-1,0]$ be a complete complex valued metric space with the metric defined by $d(x,y) = |x-y|\, e^{i\frac{\pi}{6}}$. Define

$Sx = \left[0,\frac{x}{4}\right],\ x \in [0,1]$ and $Ty = \left[0,\frac{|y|}{4}\right],\ y \in [-1,0]$

$= 0, \quad x \in [-1,0] \qquad = 0, \quad y \in [0,1]$

Now we have that

$$u(Sx,Ty) = \begin{cases} u\left(\frac{1}{4}\,|x-|y||e^{i\frac{\pi}{6}}\right), & x \in [0,1] \text{ and } y \in [-1,0] \\ 0 & otherwise \end{cases}$$

$$d(x,y) = \begin{cases} |x+y|e^{i\frac{\pi}{6}}, & x \in [0,1] \text{ and } y \in [-1,0] \\ |x-y|e^{i\frac{\pi}{6}} & otherwise \end{cases}$$

Let $\alpha = \frac{1}{2}$ and μ be positive real numbers with $\alpha + \mu < 1$.

Since $\frac{1}{4}|x-|y||e^{i\frac{\pi}{6}} \precsim |x+y|e^{i\frac{\pi}{6}}$ and $|x+y|e^{i\frac{\pi}{6}} \precsim |x+y|e^{i\frac{\pi}{6}} + \mu\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}$.

Thus, for $x,y \in X$, $u\left(\alpha|x+y|e^{i\frac{\pi}{6}} + \mu\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}\right) \subseteq u(Sx,Ty)$ is satisfied. Hence by Corollary 2.4, S and T have a unique common fixed point, $0 \in S0 \cap T0$.

E. *Corollary 2.5*

Let (X,d) be a complete complex valued metric space and let the mappings $S,T\colon X \to X$ satisfy

$$d(Sx,Ty) \precsim \alpha d(x,y) + \mu\,\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}$$

for all $x,y \in X$, where α and μ are non-negative real numbers provided $\alpha + \mu < 1$. Then S and T have a unique common fixed point.

1) *Proof:* For any two complex numbers z_1 and z_1 such that $z_1 \precsim z_2$ implies $u(z_2) \subseteq u(z_1)$. Since for $x,y \in X$, the hypothesis () is satisfied,

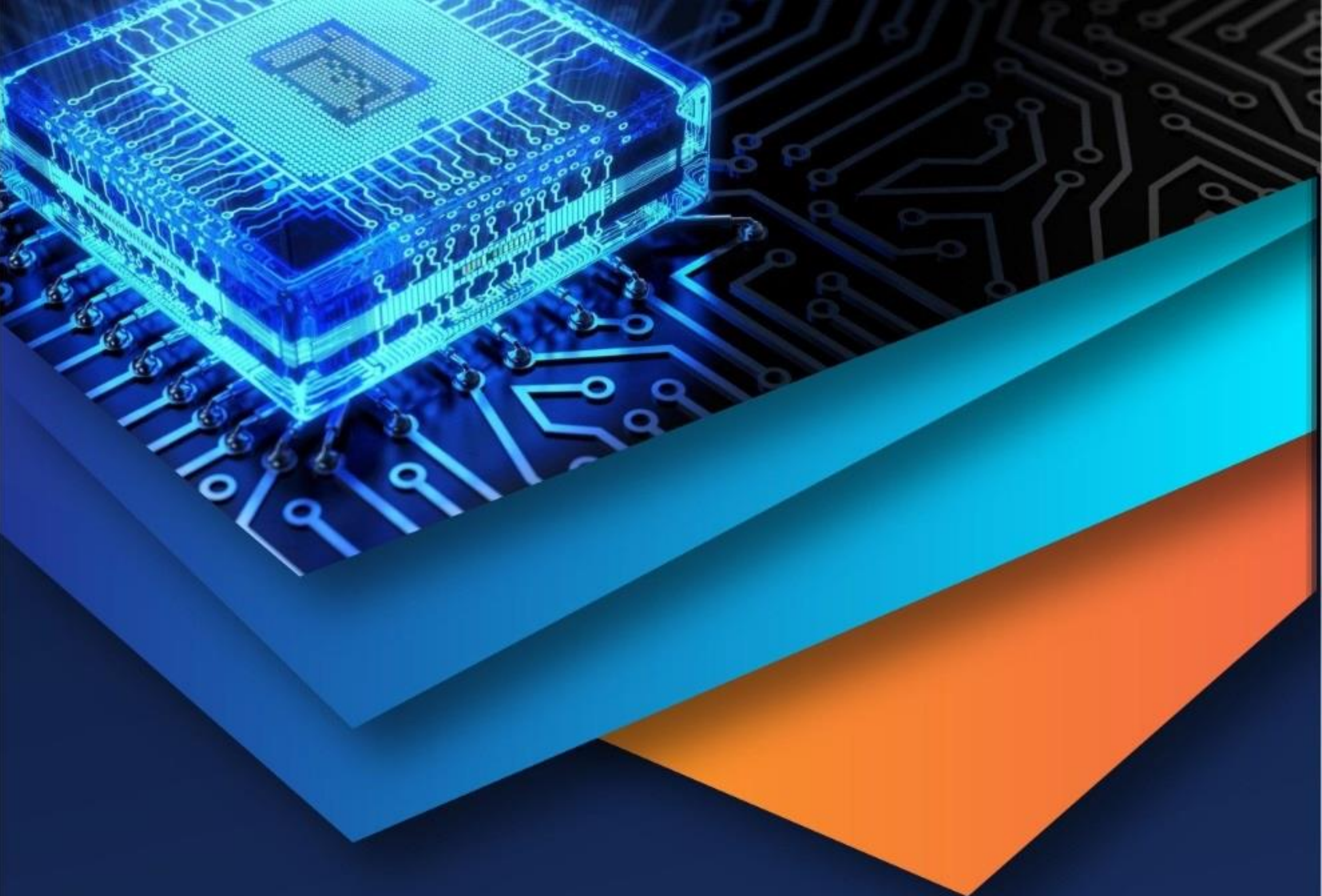
$$u(\alpha d(x,y) + \mu\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}) \subseteq u(Sx,Ty) \text{ holds.}$$

Now by Corollary 2.4, we can obtain a unique common fixed point.

REFERENCES

[1] S. B. Nadler Jr, Multivalued Contraction Mappings, Paci_c J. Math 30(2) (1969), 475
[2] Azam, A, Fisher, B, Khan, M, Common fixed point theorems in complex valued metric spaces, Numer. Funct. Anal.Optim. 32 (3) (2011), pp 243 – 25
[3] Sintunavarat ,W, Kumam, P, Generalized common fixed point theorems in complex valued metric spaces and applications, J. Inequal. Appl (2012) 2012:84
[4] Abbas, M, Arshad, M, Azam, A, Fixed points of asymptotically regular mappings in complex-valued metric spaces, Georgian Math. J., 20(2), (2013), pp213 { 22
[5] Huang, L.M, Zhang, X, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Analysis and App. 332(2), (2007) pp 1468 -1476
[6] Rouzkard, F, Imdad, M, Some common fixed point theorems on complex valued metric spaces, Computers & Mathematics with App. 64(6) (2012), pp 1866 – 1874
[7] Sintunavarat, W, Cho, Y.J, Kumam, P, Urysohn integral equations approach by com- mon fixed points in complex-valuedmetric spaces, Advances in Di_erence Equations,b 2013(2013), article 4
[8] Sitthikul, K, Saejung, S, Some fixed point theorems in complex valued metric spaces, Fixed Point Theory and Applications, 2012 (2012), article 189
[9] Verma R K, Pathak H K, Common fixed point theorems using property (E.A) in complex valued metric spaces, Thai J. Math, 11(2), (2012), pp 347 { 355
[10] Ozturk, M, Common fixed point theorems satisfying contractive type conditions in complex valued metric spaces, Abstr. Appl. Anal. 2014 (2014), Article ID 598465
[11] Ahmad J, Klin-Eam, C, Azam, A, Common Fixed Points for Multivalued Mappings in Complex Valued Metric Spaces with Applications, Abstr. Appl. Anal. 2013 (2013) 12pages Article ID 854965
[12] pings in complex valued metric spaces, Journal of inequality and App. 2013 (2013)

[13] Joshi, V, Singh, N, Singh D, ϕ-contractive multivalued mappings in complex valued metric spaces and remarks on some recent papers, Cogent Mathematics (2016), 3: 1162484.



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