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# Bipolar L-Fuzzy $\ell$-HX group and its Level sub $\ell$ HX group 

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#### Abstract

In this paper, we discussed some properties of bipolar L-fuzzy sub $\boldsymbol{\ell}-\mathrm{HX}$ group of a $\boldsymbol{\ell}-\mathrm{HX}$ group. We establish the relation between bipolar L-fuzzy sub $\boldsymbol{\ell}-\boldsymbol{H X}$ group and bipolar anti $L$-fuzzy sub $\boldsymbol{\ell}-\boldsymbol{H X}$ group. The purpose of this study is to implement the fuzzy set theory and graph theory in bipolar L-fuzzy sub $\boldsymbol{\ell}-\boldsymbol{H X}$ group. Characterizations of level subsets of a bipolar $L$ - fuzzy sub $\ell-H X$ group are given. We also discussed the relation between a bipolar $L-$ fuzzy sub $\ell-H X$ group and its level sub $\ell-H X$ groups and investigate the conditions under which a given sub $\ell-H X$ group has a properly inclusive chain of sub $\boldsymbol{\ell}$ - HX groups. In particular, we formulate how to structure an bipolar $L$ - fuzzy sub $\boldsymbol{\ell}-\boldsymbol{H X}$ group by a given chain of sub $\boldsymbol{\ell}$ HX groups.


Keywords: Bipolar L-fuzzy $\ell$ - HX group, Bipolar anti L-fuzzy $\ell-H X$ group, level subset, level sub $\ell$-HX group. AMS Subject Classification (2010): 20N25, 03E72, 03G25.

## I. INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh[9]. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld[6] gave the idea of fuzzy subgroups. In fuzzy sets the membership degree of elements range over the interval $[0,1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set and membership degree 0 indicates that an element does not belong to fuzzy set. The membership degrees on the interval $(0,1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. Li Hongxing[3] introduced the concept of HX group and the authors Luo Chengzhong, Mi Honghai, Li Hongxing[4] introduced the concept of fuzzy HX group. The author W.R.Zhang[10] commenced the concept of bipolar fuzzy sets as a generalization of fuzzy sets in 1994. K.M. Lee[2] introduced Bipolar-valued fuzzy sets and their operations. In case of Bipolar-valued fuzzy sets membership degree range is enlarged from the interval $[0,1]$ to $[-1,1]$. In a bipolar-valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the membership degree $(0,1]$ indicates that elements somewhat satisfy the property and the membership degree $[-1,0)$ indicates that elements somewhat satisfy the implicit counter-property. G.S.V.Satya Saibaba[7] initiated the study of L-fuzzy lattice ordered groups and introduced the notions of L-fuzzy sub $\ell-$ HX group. J.A Goguen[1] replaced the valuation set [0,1] by means of a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L - Fuzzy sets. R.Muthuraj, M.Sridharan[5] introduced Bipolar fuzzy HX group and its level sub HX groups. The authors K.Sunderrajan, A.Senthilkumar, R.Muthuraj [8] introduced L - fuzzy sub $\ell$-group and its level sub $\ell$-groups.

## II. PRELIMINARIES

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, $\mathrm{G}=(\mathrm{G},$.$) is a group, \mathrm{e}$ is the identity element of G , and xy , we mean $\mathrm{x} . \mathrm{y}$

## A. Definition 2.1

Let $\mu$ be a bipolar L-fuzzy subset defined on G. Let $\vartheta \subset 2^{\mathrm{G}}-\{\phi\}$ be a $\ell-$ HX group on G. A bipolar L-fuzzy set $\lambda^{\mu}$ defined on $\vartheta$ is said to be a bipolar $L$ - fuzzy sub $\ell-H X$ group on $\vartheta$ if for all $\mathrm{A}, \mathrm{B} \in \vartheta$.

1) $\left(\lambda^{\mu}\right)^{+}(\mathrm{AB}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
2) $\left(\lambda^{\mu}\right)^{-}(\mathrm{AB}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}$(B)
3) $\left(\lambda^{\mu}\right)^{+}(\mathrm{A})=\left(\lambda^{\mu}\right)^{+}\left(\mathrm{A}^{-1}\right)$
4) $\left(\lambda^{\mu}\right)^{-}(\mathrm{A})=\left(\lambda^{\mu}\right)^{-}\left(\mathrm{A}^{-1}\right)$
5) $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
6) $\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
7) $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
8) $\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}$(B)

Where $\left(\lambda^{\mu}\right)^{+}(\mathrm{A})=\max \left\{\mu^{+}(\mathrm{x}) /\right.$ for all $\left.\mathrm{x} \in \mathrm{A} \subseteq \mathrm{G}\right\}$ and
$\left(\lambda^{\mu}\right)^{-}(\mathrm{A})=\min \left\{\mu^{-}(\mathrm{x}) /\right.$ for all $\left.\mathrm{x} \in \mathrm{A} \subseteq \mathrm{G}\right\}$
B. Example 2.1

Let $\mathrm{G}=\left\{\mathrm{Z}_{5}-\{0\}, .5\right\}$ be a group and define a bipolar L- fuzzy set $\mu$ on G as $\mu^{+}(1)=0.8, \mu^{+}(2)=0.5, \mu^{+}(3)=0.5, \mu^{+}(4)=0.5$ and $\mu^{-}$ (1) $=-0.7, \mu^{-}(2)=-0.6, \mu^{-}(3)=-0.6, \mu^{-}(4)=-0.6$

By routine computations, it is easy to see that $\mu$ is a bipolar L-fuzzy sub group of G .
Let $\vartheta=\{\{1,4\},\{2,3\}\}$ be a $\ell-$ HX group of G .
Let us consider $A=\{1,4\}, B=\{2,3\}$.

| $\cdot \boldsymbol{5}$ | A | B |
| :---: | :---: | :---: |
| $\mathbf{A}$ | A | B |
| $\mathbf{B}$ | B | A |


| $\wedge$ | $\mathbf{A}$ | $\mathbf{B}$ |
| :---: | :---: | :---: |
| $\mathbf{A}$ | A | A |
| $\mathbf{B}$ | A | B |


| V | A | B |
| :---: | :---: | :---: |
| A | A | B |
| $\mathbf{B}$ | B | B |

Define $\left(\lambda^{\mu}\right)^{+}(\mathrm{A})=\max \left\{\mu^{+}(\mathrm{x}) /\right.$ for all $\left.\mathrm{x} \in \mathrm{A} \subseteq \mathrm{G}\right\}$
and
$\left(\lambda^{\mu}\right)^{-}(\mathrm{A})=\min \left\{\mu^{-}(\mathrm{x}) /\right.$ for all $\left.\mathrm{x} \in \mathrm{A} \subseteq \mathrm{G}\right\}$
Now
$\left(\lambda^{\mu}\right)^{+}(\mathrm{A})=\left(\lambda^{\mu}\right)^{+}(\{1,4\})=\max \left\{\mu^{+}(1), \mu^{+}(4)\right\}=\max \{0.8,0.5\}=0.8$
$\left(\lambda^{\mu}\right)^{+}(\mathrm{B})=\left(\lambda^{\mu}\right)^{+}(\{2,3\})=\max \left\{\mu^{+}(2), \mu^{+}(3)\right\}=\max \{0.5,0.5\}=0.5$
$\left(\lambda^{\mu}\right)^{+}(\mathrm{AB})=\left(\lambda^{\mu}\right)^{+}(\mathrm{B})=0.5$
$\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B})=\left(\lambda^{\mu}\right)^{+}(\mathrm{A})=0.8$
$\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B})=\left(\lambda^{\mu}\right)^{+}(\mathrm{B})=0.5$
$\left(\lambda^{\mu}\right)(\mathrm{A})=\left(\lambda^{\mu}\right)-(\{1,4\})=\min \left\{\mu^{-}(1), \mu^{-}(4)\right\}=\min \{-0.7,-0.6\}=-0.7$
$\left(\lambda^{\mu}\right)-(B)=\left(\lambda^{\mu}\right)-(\{2,3\})=\min \left\{\mu^{-}(2), \mu^{-}(3)\right\}=\min \{-0.6,-0.6\}=-0.6$
$\left(\lambda^{\mu}\right)-(\mathrm{AB}) \quad=\left(\lambda^{\mu}\right)-(\mathrm{B})=-0.6$
$\left(\lambda^{\mu}\right)(\mathrm{A} \wedge \mathrm{B})=\left(\lambda^{\mu}\right)(\mathrm{A})=-0.7$
$\left(\lambda^{\mu}\right)^{-( }(\mathrm{A} \vee \mathrm{B})=\left(\lambda^{\mu}\right)^{-(B)}=-0.6$
By routine computations, it is easy to see that $\lambda^{\mu}$ is a bipolar L-fuzzy sub $\ell-$ HX group of $\vartheta$.
C. Definition 2.2

Let $\mu$ be a bipolar L - fuzzy subset defined on G. Let $\vartheta \subset 2^{\mathrm{G}}-\{\phi\}$ be a $\ell-\mathrm{HX}$ group on G. A bipolar L-fuzzy set $\lambda^{\mu}$ defined on $\vartheta$ is said to be a bipolar anti $L$ - fuzzy sub $\ell-H X$ group on $\vartheta$ if for all $\mathrm{A}, \mathrm{B} \in \vartheta$.

1) $\left(\lambda^{\mu}\right)^{+}(\mathrm{AB}) \leq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
2) $\left(\lambda^{\mu}\right)^{-}(\mathrm{AB}) \geq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
3) $\left(\lambda^{\mu}\right)^{+}(\mathrm{A})=\left(\lambda^{\mu}\right)^{+}\left(\mathrm{A}^{-1}\right)$
4) $\left(\lambda^{\mu}\right)^{-}(\mathrm{A})=\left(\lambda^{\mu}\right)^{-}\left(\mathrm{A}^{-1}\right)$
5) $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
6) $\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
7) $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
8) $\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$

Where $\left(\lambda^{\mu}\right)^{+}(\mathrm{A})=\min \left\{\mu^{+}(\mathrm{x}) /\right.$ for all $\left.\mathrm{x} \in \mathrm{A} \subseteq \mathrm{G}\right\}$
and
$\left(\lambda^{\mu}\right)^{-}(\mathrm{A})=\max \left\{\mu^{-}(\mathrm{x}) /\right.$ for all $\left.\mathrm{x} \in \mathrm{A} \subseteq \mathrm{G}\right\}$

## D. Example 2.2

Let $\mathrm{G}=\left\{\mathrm{Z}_{5}-\{0\}, .{ }_{5}\right\}$ be a group and define a bipolar L - fuzzy set $\mu$ on G as $\mu^{+}(1)=0.4, \mu^{+}(2)=0.8$, $\mu^{+}(3)=0.8, \mu^{+}(4)=0.8$ and $\mu^{-}(1)=-0.5, \mu^{-}(2)=-0.7, \mu^{-}(3)=-0.7, \mu^{-}(4)=-0.7$
By routine computations, it is easy to see that $\mu$ is a bipolar anti L-fuzzy subgroup of G .
Let $\vartheta=\{\{1,4\},\{2,3\}\}$ be a $\ell-\mathrm{HX}$ group of G .
Let us consider $A=\{1,4\}, B=\{2,3\}$.

| . $\boldsymbol{5}$ | $\mathbf{A}$ | $\mathbf{B}$ |
| :---: | :---: | :---: |
| $\mathbf{A}$ | A | B |
| $\mathbf{B}$ | B | A |


| $\wedge$ | $\mathbf{A}$ | $\mathbf{B}$ |
| :---: | :---: | :---: |
| $\mathbf{A}$ | A | A |
| $\mathbf{B}$ | A | B |


| $\vee$ | $\mathbf{A}$ | $\mathbf{B}$ |
| :---: | :---: | :---: |
| $\mathbf{A}$ | A | B |
| $\mathbf{B}$ | B | B |

Define $\left(\lambda^{\mu}\right)^{+}(A)=\min \left\{\mu^{+}(x) /\right.$ for all $\left.x \in A \subseteq G\right\}$
and
$\left(\lambda^{\mu}\right)^{-}(\mathrm{A})=\max \left\{\mu^{-}(\mathrm{x}) /\right.$ for all $\left.\mathrm{x} \in \mathrm{A} \subseteq \mathrm{G}\right\}$
Now
$\left(\lambda^{\mu}\right)^{+}(\mathrm{A})=\left(\lambda^{\mu}\right)^{+}(\{1,4\})=\min \left\{\mu^{+}(1), \mu^{+}(4)\right\}=\min \{0.4,0.8\}=0.4$
$\left(\lambda^{\mu}\right)^{+}(B)=\left(\lambda^{\mu}\right)^{+}(\{2,3\})=\min \left\{\mu^{+}(2), \mu^{+}(3)\right\}=\min \{0.8,0.8\}=0.8$
$\left(\lambda^{\mu}\right)^{+}(\mathrm{AB})=\left(\lambda^{\mu}\right)^{+}(\mathrm{B})=0.8$
$\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B})=\left(\lambda^{\mu}\right)^{+}(\mathrm{A})=0.4$
$\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B})=\left(\lambda^{\mu}\right)^{+}(\mathrm{B})=0.8$
$\left(\lambda^{\mu}\right)^{-}(\mathrm{A})=\left(\lambda^{\mu}\right)^{-}(\{1,4\})=\max \left\{\mu^{-}(1), \mu^{-}(4)\right\}=\max \{-0.5,-0.7\}=-0.5$
$\left(\lambda^{\mu}\right)^{-}(\mathrm{B})=\left(\lambda^{\mu}\right)^{-}(\{2,3\})=\max \left\{\mu^{-}(2), \mu^{-}(3)\right\}=\max \{-0.7,-0.7\}=-0.7$
$\left(\lambda^{\mu}\right)^{-}(\mathrm{AB})=\left(\lambda^{\mu}\right)^{-}(\mathrm{B})=-0.7$
$\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B})=\left(\lambda^{\mu}\right)^{-}(\mathrm{A})=-0.5$
$\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B})=\left(\lambda^{\mu}\right)^{-}(\mathrm{B})=-0.7$
By routine computations, it is easy to see that $\lambda^{\mu}$ is a bipolar anti L-fuzzy sub $\ell-$ HX group of $\vartheta$.

## III. PROPERTIES OF BIPOLAR L-FUZZY SUB $\boldsymbol{\ell}$ - HX GROUP

In this section, We discuss some of the properties of bipolar L-fuzzy sub $\ell-$ HX group

## A. Definition 3.1

Let $\mu=\left(\mu^{+}, \mu^{-}\right)$and $\alpha=\left(\alpha^{+}, \alpha^{-}\right)$are bipolar L-fuzzy subsets of G. Let $\vartheta \subset 2^{\mathrm{G}}-\{\phi\}$ be a $\ell-$ HX group of G. Let $\quad \lambda^{\mu}=\left(\left(\lambda^{\mu}\right)^{+},\left(\lambda^{\mu}\right)^{-}\right)$ and $\eta^{\alpha}=\left(\left(\eta^{\alpha}\right)^{+},\left(\eta^{\alpha}\right)^{-}\right)$are bipolar L-fuzzy subsets of $\vartheta$. The union of $\lambda^{\mu}$ and $\eta^{\alpha}$ is $\quad\left(\lambda^{\mu} \cup \eta^{\alpha}\right)=\left(\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+},\left(\lambda^{\mu} \cup\right.\right.$ $\left.\left.\eta^{\alpha}\right)^{-}\right)$defined as

1) $\quad\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{A})=\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{+}(\mathrm{A})$
2) $\quad\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{A})=\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{-}(\mathrm{A})$

Where $\left(\lambda^{\mu}\right)^{+}(A)=\max \left\{\mu^{+}(x) /\right.$ for all $\left.x \in A \subseteq G\right\}$
$\left(\lambda^{\mu}\right)^{-}(A)=\min \left\{\mu^{-}(x) /\right.$ for all $\left.x \in A \subseteq G\right\}$
$\left(\eta^{\alpha}\right)^{+}(A)=\max \left\{\alpha^{+}(x) /\right.$ for all $\left.x \in A \subseteq G\right\}$
$\left(\eta^{\alpha}\right)^{-}(A)=\min \left\{\alpha^{-}(x) /\right.$ for all $\left.x \in A \subseteq G\right\}$

## B. Theorem 3.1

Let $\lambda^{\mu}$ and $\eta^{\alpha}$ be any two bipolar L-fuzzy sub $\ell-$ HX group of a $\ell-H X$ group $\vartheta$ then $\lambda^{\mu} \cup \eta^{\alpha}$ is a bipolar L-fuzzy sub $\ell-H X$ group of a $\ell$ - HX group $\vartheta$.

1) Proof
a) $\quad\left(\left(\lambda^{\mu}\right) \cup\left(\eta^{\alpha}\right)\right)^{+}(\mathrm{AB})=\left(\lambda^{\mu}\right)^{+}(\mathrm{AB}) \vee\left(\eta^{\alpha}\right)^{+}(\mathrm{AB})$

$$
\begin{aligned}
& \geq\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})\right) \vee\left(\left(\eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& =\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{+}(\mathrm{A})\right) \wedge\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{B}) \vee\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& =\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{B})
\end{aligned}
$$

$$
\begin{aligned}
& \geq\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{B}) \\
& \text { b) } \left.\quad\left(\left(\lambda^{\mu}\right) \cup\left(\eta^{\alpha}\right)\right)^{-}(\mathrm{AB})=\left(\lambda^{\mu}\right)^{-(A B}\right) \wedge\left(\eta^{\alpha}\right)^{-}(\mathrm{AB}) \\
& \leq\left(\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})\right) \wedge\left(\left(\eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{-}(\mathrm{B})\right) \\
& \left.=\left(\left(\lambda^{\mu}\right)-(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{-(\mathrm{A})}\right) \vee\left(\left(\lambda^{\mu}\right)^{-(\mathrm{B}}\right) \wedge\left(\eta^{\alpha}\right)^{-}(\mathrm{B})\right) \\
& =\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{B}) \\
& \leq\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{B})
\end{aligned}
$$

c) $\quad\left(\left(\lambda^{\mu}\right) \cup\left(\eta^{\alpha}\right)\right)^{+}\left(\mathrm{A}^{-1}\right)=\left(\lambda^{\mu}\right)^{+}\left(\mathrm{A}^{-1}\right) \vee\left(\eta^{\alpha}\right)^{+}\left(\mathrm{A}^{-1}\right)$

$$
\begin{aligned}
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{+}(\mathrm{A}) \\
& =\left(\left(\lambda^{\mu}\right) \cup\left(\eta^{\alpha}\right)\right)^{+}(\mathrm{A})
\end{aligned}
$$

d) $\quad\left(\left(\lambda^{\mu}\right) \cup\left(\eta^{\alpha}\right)\right)^{-( }\left(\mathrm{A}^{-1}\right)=\left(\lambda^{\mu}\right)\left(\mathrm{A}^{-1}\right) \wedge\left(\eta^{\alpha}\right)^{-}\left(\mathrm{A}^{-1}\right)$

$$
\begin{aligned}
& =\left(\lambda^{\mu}\right)-(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{-}(\mathrm{A}) \\
& =\left(\left(\lambda^{\mu}\right) \cup\left(\eta^{\alpha}\right)\right)^{-}(\mathrm{A})
\end{aligned}
$$

e) $\quad\left(\left(\lambda^{\mu}\right) \cup\left(\eta^{\alpha}\right)\right)^{+}(\mathrm{A} \vee \mathrm{B})=\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B}) \vee\left(\eta^{\alpha}\right)^{+}(\mathrm{A} \vee \mathrm{B})$

$$
\begin{aligned}
& \geq\left(\left(\lambda^{\mu}\right)^{+}+(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})\right) \vee\left(\left(\eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& =\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{+}(\mathrm{A})\right) \wedge\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{B}) \vee\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& =\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{B}) \\
& \geq\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{B})
\end{aligned}
$$

f) $\quad\left(\left(\lambda^{\mu}\right) \cup\left(\eta^{\alpha}\right)\right)^{-}(\mathrm{A} \vee \mathrm{B})=\left(\lambda^{\mu}\right)-(\mathrm{A} \vee \mathrm{B}) \wedge\left(\eta^{\alpha}\right)^{-}(\mathrm{A} \vee \mathrm{B})$

$$
\begin{aligned}
& \leq\left(\left(\lambda^{\mu}\right)-(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-(B)}\right) \wedge\left(\left(\eta^{\alpha}\right)^{\left.-(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{-}(\mathrm{B})\right)}\right. \\
& \left.=\left(\left(\lambda^{\mu}\right)-(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)-(\mathrm{A})\right) \vee\left(\left(\lambda^{\mu}\right)-(\mathrm{B}) \wedge\left(\eta^{\alpha}\right)^{-(\mathrm{B}}\right)\right) \\
& =\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{B}) \\
& \leq\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{B})
\end{aligned}
$$

g) $\quad\left(\left(\lambda^{\mu}\right) \cup\left(\eta^{\alpha}\right)\right)^{+}(\mathrm{A} \wedge \mathrm{B})=\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B}) \vee\left(\eta^{\alpha}\right)^{+}(\mathrm{A} \wedge \mathrm{B})$

$$
\begin{aligned}
& \geq\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})\right) \vee\left(\left(\eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& \left.=\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{+}(\mathrm{A})\right) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B}) \vee\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& =\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{B}) \\
& \geq\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{+}(\mathrm{B}) \\
& \left.\lambda^{\mu}\right)^{-( }(\mathrm{A} \wedge \mathrm{~B}) \wedge\left(\eta^{\alpha}\right)^{--}(\mathrm{A} \wedge \mathrm{~B}) \\
& \leq\left(\left(\lambda^{\mu}\right)-(\mathrm{A}) \vee\left(\lambda^{\mu}\right)-(\mathrm{B})\right) \wedge\left(\left(\eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{-}(\mathrm{B})\right) \\
& \left.=\left(\left(\lambda^{\mu}\right)-(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)-(\mathrm{A})\right) \vee\left(\left(\lambda^{\mu}\right)-(\mathrm{B}) \wedge\left(\eta^{\alpha}\right)^{-(B)}\right)\right) \\
& =\left(\lambda^{\mu} \cup \eta^{\alpha}\right)-(\mathrm{A}) \vee\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{B}) \\
& \leq\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu} \cup \eta^{\alpha}\right)^{-}(\mathrm{B})
\end{aligned}
$$

h) $\quad\left(\left(\lambda^{\mu}\right) \cup\left(\eta^{\alpha}\right)\right)^{-}(\mathrm{A} \wedge \mathrm{B})=\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B}) \wedge\left(\eta^{\alpha}\right)^{-}(\mathrm{A} \wedge \mathrm{B})$

Hence $\lambda^{\mu} \cup \eta^{\alpha}$ is a bipolar L-fuzzy sub $\ell-$ HX group of a $\ell-$ HX group $\vartheta$.
C. Definition 3.2

Let $\mu=\left(\mu^{+}, \mu^{-}\right)$and $\alpha=\left(\alpha^{+}, \alpha^{-}\right)$are bipolar L-fuzzy subsets of G. Let $\vartheta \subset 2^{\mathrm{G}}-\{\phi\}$ be a $\ell-$ HX group of G. Let $\lambda^{\mu}=\left(\left(\lambda^{\mu}\right)^{+},\left(\lambda^{\mu}\right)^{-}\right)$and $\eta^{\alpha}=\left(\left(\eta^{\alpha}\right)^{+},\left(\eta^{\alpha}\right)^{-}\right)$are bipolar L-fuzzy subsets of $\vartheta$. The intersection of $\lambda^{\mu}$ and $\eta^{\alpha}$ is $\left(\lambda^{\mu} \cap \eta^{\alpha}\right)=\left(\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+},\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-}\right)$defined as

$$
\begin{aligned}
& \left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{A})=\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{A}) \\
& \left.\left.\left.\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-( } \mathrm{A}\right)=\left(\lambda^{\mu}\right)^{-(\mathrm{A}}\right) \vee\left(\eta^{\alpha}\right)^{-( } \mathrm{A}\right)
\end{aligned}
$$

Where $\left(\lambda^{\mu}\right)^{+}(\mathrm{A})=\max \left\{\mu^{+}(\mathrm{x}) /\right.$ for all $\left.\mathrm{x} \in \mathrm{A} \subseteq \mathrm{G}\right\}$
$\left(\lambda^{\mu}\right)(\mathrm{A})=\min \left\{\mu^{-}(\mathrm{x}) /\right.$ for all $\left.\mathrm{x} \in \mathrm{A} \subseteq \mathrm{G}\right\}$
$\left(\eta^{\alpha}\right)^{+}(\mathrm{A})=\max \left\{\alpha^{+}(\mathrm{x}) /\right.$ for all $\left.\mathrm{x} \in \mathrm{A} \subseteq \mathrm{G}\right\}$
$\left(\eta^{\alpha}\right)(\mathrm{A})=\min \left\{\alpha^{-}(\mathrm{x}) /\right.$ for all $\left.\mathrm{x} \in \mathrm{A} \subseteq \mathrm{G}\right\}$

## D. Theorem 3.2

Let $\lambda^{\mu}$ and $\eta^{\alpha}$ be any two bipolar L-fuzzy sub $\ell$ - HX group of a $\ell-$ HX group $\vartheta$ then $\lambda^{\mu} \cap \eta^{\alpha}$ is a bipolar L-fuzzy sub $\ell-$ HX group of a $\ell-\mathrm{HX}$ group $\vartheta$.

1) Proof
a) $\quad\left(\left(\lambda^{\mu}\right) \cap\left(\eta^{\alpha}\right)\right)^{+}(\mathrm{AB})=\left(\lambda^{\mu}\right)^{+}(\mathrm{AB}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{AB})$

$$
\begin{aligned}
& \geq\left(\left(\lambda^{\mu}\right)^{+}+(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})\right) \wedge\left(\left(\eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& =\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{A})\right) \wedge\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{B}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& =\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{B}) \\
& \geq\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{B})
\end{aligned}
$$

b) $\left.\quad\left(\left(\lambda^{\mu}\right) \cap\left(\eta^{\alpha}\right)\right)^{-}(\mathrm{AB})=\left(\lambda^{\mu}\right)^{-(A B}\right) \vee\left(\eta^{\alpha}\right)^{-}(\mathrm{AB})$

$$
\begin{aligned}
& \left.\leq\left(\left(\lambda^{\mu}\right)-(\mathrm{A}) \vee\left(\lambda^{\mu}\right)-(\mathrm{B})\right) \vee\left(\left(\eta^{\alpha}\right)^{-( }(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{-(\mathrm{B}}\right)\right) \\
& =\left(\left(\lambda^{\mu}\right)^{\left.-(\mathrm{A}) \vee\left(\eta^{\alpha}\right)-(\mathrm{A})\right) \vee\left(\left(\lambda^{\mu}\right)-(\mathrm{B}) \vee\left(\eta^{\alpha}\right)^{-(B)}\right)}\right. \\
& =\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-(B)} \\
& \leq\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-(A)} \vee\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-(B)}
\end{aligned}
$$

c) $\quad\left(\left(\lambda^{\mu}\right) \cap\left(\eta^{\alpha}\right)\right)^{+}\left(\mathrm{A}^{-1}\right)=\left(\lambda^{\mu}\right)^{+}\left(\mathrm{A}^{-1}\right) \wedge\left(\eta^{\alpha}\right)^{+}\left(\mathrm{A}^{-1}\right)$

$$
\begin{aligned}
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{A}) \\
& =\left(\left(\lambda^{\mu}\right) \cap\left(\eta^{\alpha}\right)\right)^{+}(\mathrm{A})
\end{aligned}
$$

d) $\quad\left(\left(\lambda^{\mu}\right) \cap\left(\eta^{\alpha}\right)\right)^{-( }\left(\mathrm{A}^{-1}\right)=\left(\lambda^{\mu}\right)\left(\mathrm{A}^{-1}\right) \vee\left(\eta^{\alpha}\right)^{-}\left(\mathrm{A}^{-1}\right)$

$$
\begin{aligned}
& =\left(\lambda^{\mu}\right)^{-(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{-}(\mathrm{A})} \\
& =\left(\left(\lambda^{\mu}\right) \cap\left(\eta^{\alpha}\right)\right)^{-}(\mathrm{A})
\end{aligned}
$$

e) $\quad\left(\left(\lambda^{\mu}\right) \cap\left(\eta^{\alpha}\right)\right)^{+}(\mathrm{A} \vee \mathrm{B})=\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{A} \vee \mathrm{B})$

$$
\begin{aligned}
& \geq\left(\left(\lambda^{\mu}\right)^{+}+(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})\right) \wedge\left(\left(\eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& =\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{A})\right) \wedge\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{B}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& =\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{B}) \\
& \geq\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{B})
\end{aligned}
$$

f) $\quad\left(\left(\lambda^{\mu}\right) \cap\left(\eta^{\alpha}\right)\right)^{-}(\mathrm{A} \vee \mathrm{B})=\left(\lambda^{\mu}\right)-(\mathrm{A} \vee \mathrm{B}) \vee\left(\eta^{\alpha}\right)^{-}(\mathrm{A} \vee \mathrm{B})$

$$
\begin{aligned}
& \leq\left(\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)-(\mathrm{B})\right) \vee\left(\left(\eta^{\alpha}\right)^{--}(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{-}(\mathrm{B})\right) \\
& =\left(\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\eta^{\alpha}\right)-(\mathrm{A})\right) \vee\left(\left(\lambda^{\mu}\right)-(\mathrm{B}) \vee\left(\eta^{\alpha}\right)^{-}(\mathrm{B})\right) \\
& =\left(\lambda^{\mu} \cap \eta^{\alpha}\right)-(\mathrm{A}) \vee\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-}(\mathrm{B}) \\
& \leq\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-}-(\mathrm{B}) \\
& \left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{~B}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{A} \wedge \mathrm{~B}) \\
& \geq\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})\right) \wedge\left(\left(\eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& =\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{A})\right) \wedge\left(\left(\lambda^{\mu}\right)^{+}(\mathrm{B}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{B})\right) \\
& =\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{B}) \\
& \geq\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{+}(\mathrm{B})
\end{aligned}
$$

g) $\quad\left(\left(\lambda^{\mu}\right) \cap\left(\eta^{\alpha}\right)\right)^{+}(\mathrm{A} \wedge \mathrm{B})=\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B}) \wedge\left(\eta^{\alpha}\right)^{+}(\mathrm{A} \wedge \mathrm{B})$
h) $\quad\left(\left(\lambda^{\mu}\right) \cap\left(\eta^{\alpha}\right)\right)^{-}(\mathrm{A} \wedge \mathrm{B})=\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B}) \vee\left(\eta^{\alpha}\right)^{-}(\mathrm{A} \wedge \mathrm{B})$

$$
\begin{aligned}
& \leq\left(\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})\right) \vee\left(\left(\eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{-}(\mathrm{B})\right) \\
& \left.=\left(\left(\lambda^{\mu}\right)-(\mathrm{A}) \vee\left(\eta^{\alpha}\right)^{-(\mathrm{A}}\right)\right) \vee\left(\left(\lambda^{\mu}\right)-(\mathrm{B}) \vee\left(\eta^{\alpha}\right)^{-(\mathrm{B})}\right) \\
& =\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-}(\mathrm{B}) \\
& \leq\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu} \cap \eta^{\alpha}\right)^{-} \text {(B) }
\end{aligned}
$$

Hence $\lambda^{\mu} \cap \eta^{\alpha}$ is a bipolar L-fuzzy sub $\ell$ - HX group of a $\ell-$ HX group $\vartheta$.

## E. Theorem 3.3

Let $\lambda^{\mu}$ be a bipolar L-fuzzy sub $\ell-$ HX group of a $\ell-$ HX group $\vartheta$ if and only if $\left(\lambda^{\mu}\right)^{C}$ is a bipolar anti L-fuzzy sub $\ell-$ HX group of a $\ell$ - HX group $\vartheta$.

1) Proof: Let $\lambda^{\mu}$ be a bipolar L-fuzzy sub $\ell-$ HX group of a $\ell-\mathrm{HX}$ group $\vartheta$ if for $\mathrm{A}, \mathrm{B} \in \vartheta$, we have
a) $\quad\left(\lambda^{\mu}\right)^{+}(\mathrm{AB}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
$\Leftrightarrow 1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{AB}) \geq\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A})\right) \wedge\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{B})\right)$
$\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{AB}) \leq 1-\left(\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A})\right) \wedge\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{B})\right)\right.$
$\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{AB}) \leq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
b) $\quad\left(\lambda^{\mu}\right)-(\mathrm{AB}) \leq\left(\lambda^{\mu}\right)(\mathrm{A}) \vee\left(\lambda^{\mu}\right)-(\mathrm{B})$
```
    \(\Leftrightarrow 1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{AB}) \leq\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A})\right) \vee\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{B})\right)\)
    \(\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{AB}) \geq 1-\left(\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A})\right) \vee\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{B})\right)\right.\)
    \(\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{AB}) \geq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{-}(\mathrm{B})\)
c) \(\quad\left(\lambda^{\mu}\right)^{+}\left(\mathrm{A}^{-1}\right)=\left(\lambda^{\mu}\right)^{+}(\mathrm{A})\)
    \(\Leftrightarrow 1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}\left(\mathrm{A}^{-1}\right)=1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A})\)
    \(\left.\left.\Leftrightarrow\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}\left(\mathrm{A}^{-1}\right)=\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A})\)
d) \(\quad\left(\lambda^{\mu}\right)^{-}\left(\mathrm{A}^{-1}\right)=\left(\lambda^{\mu}\right)^{-}(\mathrm{A})\)
        \(\Leftrightarrow 1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}\left(\mathrm{A}^{-1}\right)=1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A})\)
        \(\left.\left.\Leftrightarrow\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}\left(\mathrm{A}^{-1}\right)=\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A})\)
e)
    \(\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})\)
    \(\Leftrightarrow 1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A} \vee \mathrm{B}) \geq\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A})\right) \wedge\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{B})\right)\)
    \(\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A} \vee \mathrm{B}) \leq 1-\left(\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A})\right) \wedge\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{B})\right)\right.\)
    \(\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A} \vee \mathrm{B}) \leq\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A}) \vee\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{B})\)
f) \(\quad\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})\)
    \(\Leftrightarrow 1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A} \vee \mathrm{B}) \leq\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A})\right) \vee\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{B})\right)\)
    \(\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A} \vee \mathrm{B}) \geq 1-\left(\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A})\right) \vee\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{B})\right)\right.\)
    \(\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A} \vee \mathrm{B}) \geq\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A}) \wedge\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{B})\)
    \(\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})\)
    \(\Leftrightarrow 1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A} \wedge \mathrm{B}) \geq\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A})\right) \wedge\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{B})\right)\)
    \(\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A} \wedge \mathrm{B}) \leq 1-\left(\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A})\right) \wedge\left(1-\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{B})\right)\right.\)
    \(\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A} \wedge \mathrm{B}) \leq\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{A}) \vee\left(\left(\lambda^{\mu}\right)^{+}\right)^{\mathrm{C}}(\mathrm{B})\)
h) \(\quad\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})\)
    \(\Leftrightarrow 1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A} \wedge \mathrm{B}) \leq\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A})\right) \vee\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{B})\right)\)
    \(\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A} \wedge \mathrm{B}) \geq 1-\left(\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A})\right) \vee\left(1-\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{B})\right)\right.\)
    \(\Leftrightarrow\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A} \wedge \mathrm{B}) \geq\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{A}) \wedge\left(\left(\lambda^{\mu}\right)^{-}\right)^{\mathrm{C}}(\mathrm{B})\)
    Hence \(\left(\lambda^{\mu}\right)^{C}\) is a Bipolar anti L-fuzzy sub \(\ell-\) HX group of a \(\ell-\) HX group \(\vartheta\).
```


## IV. PROPERTIES OF LEVEL SUBSETS OF A BIPOLAR L-FUZZY SUB $\boldsymbol{\ell}$ - HX GROUP

In this section, We introduce the concept of level subsets of a bipolar L-fuzzy sub $\ell-$ HX group and discuss some of its properties.

## A. Definition 4.1

Let $\lambda^{\mu}$ be a bipolar L-fuzzy sub $\ell-$ HX group of a $\ell-$ HX group $\vartheta$. For any $<\alpha, \beta>\in[0,1] x[-1,0]$, We define the set $\lambda^{\mu}<\alpha, \beta \gg\{$ $A \in \vartheta /\left(\lambda^{\mu}\right)^{+}(A) \geq \alpha$ and $\left.\left(\lambda^{\mu}\right)^{-}(B) \leq \beta\right\}$ is called the $<\alpha, \beta>$ level subset of $\lambda^{\mu}$ or simply the level subset of $\lambda^{\mu}$.
B. Theorem 4.1

Let $\lambda^{\mu}$ be a bipolar L-fuzzy sub $\ell-H X$ group of a $\ell-H X$ group $\vartheta$ then for $<\alpha, \beta>\in[0,1] x[-1,0]$ such that $\left(\lambda^{\mu}\right)^{+}(E) \geq \alpha,\left(\lambda^{\mu}\right)^{-}(\mathrm{E}) \leq \beta$ and $\lambda^{\mu}{ }_{<\alpha, \beta>}$ is a sub $\ell-$ HX group of $\vartheta$.

1) Proof: For all $\mathrm{A}, \mathrm{B} \in \lambda^{\mu}<\alpha, \beta>$ we have $\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \geq \alpha,\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \leq \beta$ and $\left(\lambda^{\mu}\right)^{+}(\mathrm{B}) \geq \alpha,\left(\lambda^{\mu}\right)^{-}(\mathrm{B}) \leq \beta$

Now $\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$

$$
\begin{aligned}
& \geq \alpha \wedge \alpha \\
& =\alpha
\end{aligned}
$$

$\Rightarrow \quad\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right) \geq \alpha$ $\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\beta)$

$$
\leq \beta \vee \beta
$$

$$
=\beta
$$

$\Rightarrow \quad\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right) \leq \beta$
Hence, $\mathrm{AB}^{-1} \in \lambda^{\mu}<\alpha, \beta>$
Now $\left(\lambda^{\mu}\right)^{+}(A \vee B) \geq\left(\lambda^{\mu}\right)^{+}(A) \wedge\left(\lambda^{\mu}\right)^{+}(B)$

$$
\geq \alpha \wedge \alpha
$$

```
        \(=\alpha\)
\(\Rightarrow \quad\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B}) \geq \alpha\)
    \(\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})\)
            \(\leq \beta \vee \beta\)
        \(=\beta\)
\(\Rightarrow \quad\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B}) \leq \beta\)
```

Hence, $\mathrm{A} \vee \mathrm{B} \in \lambda^{\mu}{ }_{<\alpha, \beta>}$
Now $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$

$$
\geq \alpha \wedge \alpha
$$

$$
=\alpha
$$

$$
\Rightarrow \quad\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{~B}) \geq \alpha
$$

$$
\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{~B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
$$

$$
\leq \beta \vee \beta
$$

$$
=\beta
$$

$\Rightarrow \quad\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B}) \leq \beta$
Hence, $\mathrm{A} \wedge \mathrm{B} \in \lambda^{\mu}{ }_{<\alpha, \beta>}$
Hence, $\lambda^{\mu}{ }_{<\alpha, \beta>}$ is a $\operatorname{sub} \ell-$ HX group of $\vartheta$.
C. Definition 4.2

Let $\lambda^{\mu}$ is a bipolar L-fuzzy sub $\ell-$ HX group of a $\ell-$ HX group $\vartheta$. The sub $\ell-$ HX groups $\lambda^{\mu}{ }_{<\alpha, \beta>}$ for $<\alpha, \beta>\in[0,1] \times[-1,0]$ and $\left(\lambda^{\mu}\right)^{+}(E) \geq \alpha,\left(\lambda^{\mu}\right)^{-}(E) \leq \beta$ are called level sub $\ell-H X$ groups of $\lambda^{\mu}$.

## D. Theorem 4.2

Let $\vartheta$ be a $\ell-$ HX group and $\lambda^{\mu}$ be a bipolar L-fuzzy subset of $\vartheta$ such that $\lambda^{\mu}{ }_{<\alpha, \beta>}$ is a sub $\ell-H X$ group of $\vartheta$ for $<\alpha, \beta>\in[0,1]$ $\mathrm{x}[-1,0]$ such that $\left(\lambda^{\mu}\right)^{+}(\mathrm{E}) \geq \alpha,\left(\lambda^{\mu}\right)^{-}(\mathrm{E}) \leq \beta$.Then $\lambda^{\mu}$ is a bipolar L-fuzzy sub $\ell-$ HX group of $\vartheta$.

1) Proof: Let $A, B \in \vartheta$, let $A \in \lambda^{\mu}{ }_{<\alpha 1, \beta 1>} \Rightarrow\left(\lambda^{\mu}\right)^{+}(A)=\alpha_{1},\left(\lambda^{\mu}\right)^{-}(A)=\beta_{1}$ and $B \in \lambda^{\mu}{ }_{<\alpha 2, \beta 2>} \Rightarrow\left(\lambda^{\mu}\right)^{+}(B)=\alpha_{2},\left(\lambda^{\mu}\right)^{-}(B)=\beta_{2}$

Suppose $\lambda^{\mu}{ }_{<\alpha 1, \beta 1>}<\lambda^{\mu}{ }_{<\alpha 2, \beta 2>}$ then $A, B \in \lambda^{\mu}{ }_{<\alpha 2, \beta 2>}$ As $\lambda^{\mu}{ }_{<\alpha 2, \beta 2>}$ is a sub $\ell-H X$ group of $\vartheta, A^{-1} \in \lambda^{\mu}{ }_{<\alpha 2, \beta 2>}$,
$A \wedge B \in \lambda^{\mu}{ }_{<\alpha 2, \beta 2>}$ and $A \vee B \in \lambda^{\mu}{ }_{<\alpha 2, \beta 2>}$,
Now $\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right) \quad \geq \alpha_{2}$

$$
\begin{aligned}
& =\alpha_{1} \wedge \alpha_{2} \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right) \quad \leq \beta_{2}$

$$
\begin{aligned}
& =\beta_{1} \vee \beta_{2} \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{+}(A \vee B) \geq \alpha_{2}$

$$
\begin{aligned}
= & \alpha_{1} \wedge \alpha_{2} \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}(A \vee B) \leq \beta_{2}$

$$
\begin{aligned}
& =\beta_{1} \vee \beta_{2} \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Now $\left(\lambda^{\mu}\right)^{+}(A \wedge B) \geq \alpha_{2}$

$$
\begin{aligned}
= & \alpha_{1} \wedge \alpha_{2} \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B}) \leq \beta_{2}$

$$
\begin{aligned}
& =\beta_{1} \vee \beta_{2} \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)-(\mathrm{A} \wedge \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Hence $\lambda^{\mu}$ is a bipolar L-fuzzy sub $\ell-$ HX group of $\vartheta$.

## E. Theorem 4.3

Let $\vartheta$ be a $\ell-$ HX group and $\lambda^{\mu}$ be a bipolar L-fuzzy sub $\ell-$ HX group of $\vartheta$. If two bipolar level sub $\ell-$ HX groups $\left.\lambda^{\mu}<\alpha, \gamma\right\rangle, \lambda^{\mu}{ }_{<\beta, \delta>}$ with $\alpha<\beta$ and $\delta<\gamma$ of $\lambda^{\mu}$ are equal if and only if there is no $A \in \vartheta$ such that $\alpha \leq\left(\lambda^{\mu}\right)^{+}(\mathrm{A})<\beta$ and $\delta<\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \leq \gamma$

1) Proof: Let $\lambda^{\mu}{ }_{<\alpha, \gamma>}=\lambda^{\mu}{ }_{\beta \beta, \delta>}$. Suppose that there exists $A \in \vartheta$ such that $\alpha \leq\left(\lambda^{\mu}\right)^{+}(\mathrm{A})<\beta$ and $\delta<\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \leq \gamma$

Then $\lambda^{\mu}{ }_{<\beta, \delta>} \subset \lambda^{\mu}<\alpha, \gamma>$ since $A \in \lambda^{\mu}<\alpha, \gamma>$ but not in $\lambda^{\mu}{ }_{\beta \beta, \delta>}$ which contradicts the hypothesis. Hence there exists no $A \in \vartheta$ such that $\alpha \leq\left(\lambda^{\mu}\right)^{+}(\mathrm{A})<\beta$ and $\delta<\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \leq \gamma$
Conversely, let there be no $\mathrm{A} \in \vartheta$ such that $\alpha \leq\left(\lambda^{\mu}\right)^{+}(\mathrm{A})<\beta$ and $\delta<\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \leq \gamma$
Since $\alpha<\beta$ and $\delta<\gamma$ of $\lambda^{\mu}$ we have $\lambda^{\mu}{ }_{<\beta, \delta>} \subset \lambda^{\mu}{ }_{<\alpha, \gamma}$
Let $A \in \lambda^{\mu}<\alpha, \gamma$ then $\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \geq \alpha$ and $\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \leq \gamma$ since there exists no $\mathrm{A} \in \vartheta$ such that $\alpha \leq\left(\lambda^{\mu}\right)^{+}(\mathrm{A})<\beta$ and $\delta<\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \leq \gamma$, we have $\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \geq \beta$ and $\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \leq \delta$ which implies $\mathrm{A} \in \lambda^{\mu}{ }_{<\beta, \delta>}$ that is $\lambda^{\mu}{ }_{<\alpha, \gamma>} \subset \lambda^{\mu}{ }^{\langle\beta, \delta>}$. Hence $\lambda^{\mu}{ }_{\langle\alpha, \gamma}=\lambda^{\mu}<\beta, \delta>$.
F. Theorem 4.4

A L-fuzzy subset $\lambda^{\mu}$ of $\vartheta$ is a bipolar L-fuzzy sub $\ell-$ HX group of $\vartheta$ if and only if the level subsets $\left.\lambda^{\mu}<\alpha, \beta\right\rangle$, $\langle\alpha, \beta\rangle \in$ Image $\lambda^{\mu}$ are sub $\ell-$ HX group of $\vartheta$.

1) Proof: It is clear.
G. Theorem 4.5

Any sub $\ell$ - HX group H of a $\ell-$ HX group $\vartheta$ can be realized as a level sub $\ell-$ HX group of some bipolar L-fuzzy sub $\ell-$ HX group of $\vartheta$.

1) Proof.: Let $\lambda^{\mu}=\left(\left(\lambda^{\mu}\right)^{+},\left(\lambda^{\mu}\right)^{-}\right)$be a bipolar L-fuzzy subset and $\mathrm{A} \in \vartheta$,
2) Define $\left(\lambda^{\mu}\right)^{+}\{(\bar{A})=\alpha$, if $A \in H$

$$
\{0 \text {, if } \mathrm{A} \notin \mathrm{H} \text { and }
$$

$\left(\lambda^{\mu}\right)^{-}(\mathrm{A})=\left\{\begin{array}{l}0, \text { if } \mathrm{A} \in \mathrm{H} \\ \beta, \text { if } \mathrm{A} \notin \mathrm{H}\end{array}\right.$
we shall prove $\lambda^{\mu}$ be a bipolar L-fuzzy sub $\ell-$ HX group of $\vartheta$.
Let $\mathrm{A}, \mathrm{B} \in \mathcal{\vartheta}$
i) Suppose $A, B \in H$ then $A B \in H, A B^{-1} \in H, A \wedge B \in H$ and $A \vee B \in H$
$\left(\lambda^{\mu}\right)^{+}(\mathrm{A})=\left(\lambda^{\mu}\right)^{+}(\mathrm{B})=\alpha$ and $\left(\lambda^{\mu}\right)^{-}(\mathrm{A})=\left(\lambda^{\mu}\right)^{-}(\mathrm{B})=0$
Now $\quad\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right)=\alpha$

$$
\begin{aligned}
& \geq \alpha \wedge \alpha \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right)=0$ $\leq 0 \vee 0$

$$
=\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
$$

Hence, $\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Now $\left(\lambda^{\mu}\right)^{+}(A \vee B)=\alpha$

$$
\begin{aligned}
& \geq \alpha \wedge \alpha \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B}) \geq\left(\lambda^{\mathrm{H}}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B})=0$

$$
\leq 0 \vee 0
$$

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$$
=\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
$$

Hence, $\left(\lambda^{\mu}\right)^{-}(A \vee B) \leq\left(\lambda^{\mu}\right)^{-}(A) \vee\left(\lambda^{\mu}\right)^{-}(B)$
Now $\quad\left(\lambda^{\mu}\right)^{+}(A \wedge B)=\alpha$

$$
\begin{aligned}
& \geq \alpha \wedge \alpha \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}(A \wedge B)=0$

$$
\begin{aligned}
& \leq 0 \vee 0 \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\quad\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Suppose $A \in H, B \notin H$ then $A B \notin H, A B^{-1} \notin H, A \wedge B \in H$ or $A \wedge B \notin H$ and $A \vee B \in H$ or $A \vee B \notin H$
$\left(\lambda^{\mu}\right)^{+}(A)=\alpha,\left(\lambda^{\mu}\right)^{+}(B)=0$ and $\left(\lambda^{\mu}\right)^{-}(A)=0,\left(\lambda^{\mu}\right)^{-}(B)=\beta$
3) Define
a) $\left(\lambda^{\mu}\right)^{+}(A \wedge B)=\alpha$, if $\begin{aligned} & A \wedge B \in H \\ & 0, \text { if } A \wedge B \notin H\end{aligned}$
and

$$
\left(\lambda^{\mu}\right)^{-}(A \wedge B)=\left\{\begin{array}{l}
0, \text { if } A \wedge B \in H \\
\beta, \text { if } A \wedge B \notin H
\end{array}\right.
$$

b) $\left(\lambda^{\mu}\right)^{+}(A \vee B)=\alpha,\left\{\begin{array}{l}\text { if } A \vee B \in H \\ 0, \text { if } A \vee B \notin H\end{array}\right.$ and

$$
\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{~B})=\left\{\begin{array}{l}
0, \text { if } \mathrm{A} \vee \mathrm{~B} \in \mathrm{H} \\
\beta, \text { if } \mathrm{A} \vee \mathrm{~B} \notin \mathrm{H}
\end{array}\right.
$$

Let $A \wedge B \in H$ and $A \vee B \in H$
Now $\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right)=\alpha$

$$
\begin{aligned}
& \geq \alpha \wedge 0 \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right)=\beta$

$$
\begin{aligned}
& \leq 0 \vee \beta \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Now $\left(\lambda^{\mu}\right)^{+}(A \vee B)=\alpha$

$$
\begin{aligned}
& \geq \alpha \wedge 0 \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}(A \vee B)=0$

$$
\begin{aligned}
& \leq 0 \vee \beta \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{+}(A \wedge B)=\alpha$

$$
\begin{aligned}
& \geq \alpha \wedge 0 \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}(A \wedge B)=0$

$$
\begin{aligned}
& \leq 0 \vee \beta \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\quad\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Let $\mathrm{A} \wedge \mathrm{B} \notin \mathrm{H}$ and $\mathrm{A} \vee \mathrm{B} \notin \mathrm{H}$
Now $\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right)=\alpha$

$$
\begin{aligned}
& \geq \alpha \wedge 0 \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right)=\beta$

$$
\begin{aligned}
& \leq 0 \vee \beta \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Now $\left(\lambda^{\mu}\right)^{+}(A \vee B)=0$

$$
\begin{aligned}
& \geq \alpha \wedge 0 \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}(A \vee B)=\beta$

$$
\begin{aligned}
& \leq 0 \vee \beta \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{+}(A \wedge B)=0$

$$
\begin{aligned}
& \geq \alpha \wedge 0 \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)-(A \wedge B)=\beta$

$$
\begin{aligned}
& \leq 0 \vee \beta \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\quad\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
iii)suppose $A, B \notin H$ then $A^{-1} \in H$ or $A B^{-1} \notin H, A \wedge B \notin H$ and $A \vee B \notin H$
$\left(\lambda^{\mu}\right)^{+}(A)=\left(\lambda^{\mu}\right)^{+}(B)=0,\left(\lambda^{\mu}\right)^{-}(A)=\left(\lambda^{\mu}\right)^{-}(B)=\beta,\left(\lambda^{\mu}\right)^{+}(A \wedge B)=\left(\lambda^{\mu}\right)^{+}(A \vee B)=0$ and $\left(\lambda^{\mu}\right)^{-}(A \wedge B)=\left(\lambda^{\mu}\right)^{-}(A \vee B)=\beta$

$$
\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right)=\left\{\begin{array}{c}
\alpha, \text { if } \mathrm{AB}^{-1} \in \mathrm{H} \\
0, \text { if } \mathrm{AB}^{-1} \notin \mathrm{H}
\end{array} \quad\right. \text { and }
$$

## Let $\mathrm{AB}^{-1} \notin \mathrm{H}$

$$
\begin{aligned}
& \left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right)=\left\{\begin{array}{l}
0, \text { if } \mathrm{AB}^{-1} \in \mathrm{H} \\
\beta, \text { if } \mathrm{AB}^{-1} \notin \mathrm{H}
\end{array}\right. \\
& \notin \mathrm{H}
\end{aligned}
$$

Now $\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right)=0$

$$
\begin{aligned}
& \geq 0 \wedge 0 \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}\left(\mathrm{AB}^{-1}\right) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right)=\beta$

$$
\begin{aligned}
& \leq \beta \vee \beta \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{-}\left(\mathrm{AB}^{-1}\right) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Now $\left(\lambda^{\mu}\right)^{+}(A \vee B)=0$

$$
\begin{aligned}
& \geq 0 \wedge 0 \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \vee \mathrm{B}) \geq\left(\lambda^{\mathrm{H}}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B})=\beta$

$$
\leq \beta \vee \beta
$$

$$
=\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
$$

Hence, $\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \vee \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B})=0$

$$
\begin{aligned}
& \geq 0 \wedge 0 \\
& =\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})
\end{aligned}
$$

Hence, $\left(\lambda^{\mu}\right)^{+}(\mathrm{A} \wedge \mathrm{B}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A}) \wedge\left(\lambda^{\mu}\right)^{+}(\mathrm{B})$
Now $\quad\left(\lambda^{\mu}\right)^{-}(A \wedge B)=\beta$

$$
\begin{aligned}
& \leq \beta \vee \beta \\
& =\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})
\end{aligned}
$$

Hence, $\quad\left(\lambda^{\mu}\right)^{-}(\mathrm{A} \wedge \mathrm{B}) \leq\left(\lambda^{\mu}\right)^{-}(\mathrm{A}) \vee\left(\lambda^{\mu}\right)^{-}(\mathrm{B})$
Thus in all cases, $\lambda^{\mu}$ be a bipolar L-fuzzy sub $\ell-$ HX group of $\vartheta$. For this bipolar L-fuzzy sub $\ell-$ HX group,
$\lambda^{\mu}\langle\alpha, \beta\rangle=H$
4) Remark: As a Consequence of the Theorem 4.3, Theorem 4.4 the level sub $\ell-$ HX groups of a bipolar L-fuzzy sub $\ell-$ HX group $\lambda^{\mu}$ of a $\ell-$ HX group $\vartheta$ form a chain. Since $\left(\lambda^{\mu}\right)^{+}(\mathrm{E}) \geq\left(\lambda^{\mu}\right)^{+}(\mathrm{A})$ and $\left(\lambda^{\mu}\right)^{-}(\mathrm{E}) \leq\left(\lambda^{\mu}\right)^{-}$(A) for all A in $\vartheta$
Therefore, $\left.\lambda^{\mu}{ }_{<a 0}, \beta 0\right\rangle, \alpha \in[0,1]$ and $\beta \in[-1,0]$
Where $\left(\lambda^{\mu}\right)^{+}(\mathrm{E})=\alpha_{0},\left(\lambda^{\mu}\right)^{-}(\mathrm{E})=\beta_{0}$ is the smallest sub $\ell-\mathrm{HX}$ group and
we have the chain $\{E\} \subseteq \lambda^{\mu}{ }_{<\alpha 0} 0, \beta>\subseteq \lambda^{\mu}{ }_{<\alpha 1, \beta 1>} \subseteq \lambda^{\mu}{ }_{<\alpha 2, \beta 2>} \subseteq \ldots \ldots . . \subseteq \lambda^{\mu}{ }_{<\alpha n, \beta n>}=\vartheta$,
where $\alpha_{0}>\alpha_{1}>\alpha_{2}>\ldots \ldots>\alpha_{n}$ and $\beta_{0}<\beta_{1}<\beta_{2}<\ldots \ldots . .<\beta_{n}$

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