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Forbidden 3-Colored Posets of Cover-Incomparable Line Graphs

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Abstract: The cover-incomparability graph of a poset P is the edge-union of the covering and the incomparability graph of P. As a continuation of the study 3-colored diagrams we characterize some forbidden ⊲ - preserving subposets of the posets whose cover-in comparability graphs are not line graphs is proved.

Index Terms: Cover-incomparability graph, Line graph, Poset.

I. INTRODUCTION AND PRELIMINARIES

Cover-incomparability graphs of posets, or shortly C-I graphs, were introduced in [2] as the underlying graphs of the standard interval function or transit function on posets (for more on transit functions in discrete structures .[3, 4, 5, 6, 11]). On the other hand, C-I graphs can be defined as the edge-union of the covering and incomparability graph of a poset; in fact, they present the only non-trivial way to obtain an associated graph as unions and/or intersections of the edge sets of the three standard associated graphs (i.e. covering, comparability and incomparability graph). In the paper that followed [9], it was shown that the complexity of recognizing whether a given graph is the C-I graph of some poset is in general NP-complete. In [1] the problem was investigated for the classes of split graphs and block graphs, and the C-I graphs within these two classes of graphs were characterized. This resulted in linear-time recognition algorithms for C-I block and C-I split graphs. It was also shown in [1] that whenever a C-I graph is a chordal graph, it is necessarily an interval graph, however a structural characterization of C-I interval graphs (and thus C-I chordal graphs) is still open. C-I distance-hereditary graphs have been characterized and shown to be efficiently recognizable [10].

Let P = (V;_≤) be a poset. If u ≤ v but u ≠ v, then we write u < v. For u, v ϵ V we say that v *covers* u in P if u < v and there is no w in V with u < w < v. If u ≤ v we will sometimes say that u is *below* v, and that v is *above* u. Also, we will write u ⊲ v if v covers u; and u ⊲⊲v if u is below v but not covered by v. By u || v we denote that u and v are incomparable. Let V' be a nonempty subset of V . Thenthere is a natural poset Q = (V';≤ ′), where u ≤ ′ v if and only if u ≤ v for any u,v$\epsilon V'$. The poset Q is called a *subposet* of P and its notation is simplified to Q = (V';≤). If, in addition, together with any two comparable elements u and v of Q, a chain of shortest length between u and v of P is also in Q, we say that Q is an isometric sub poset of P. Recall that a poset P is *dual* to a poset Q if for any x, y ϵ P the following holds: x ≤ y in P if and only if y ≤ x in Q. Given a poset P, its cover-incomparability graph G_P has V as its vertex set, and uv is an edge of G_P if u ⊲v, v ⊲ u, or u and v are incomparable. A graph that is a cover-incomparability graph of some poset P will be called a C-I graph.

Lemma 1 **[**2**]** Let P be a poset and G_P its C-I graph. Then
(i). G_P is connected;
(ii). vertices in an independent set of G_P lie on a common chain of P;
(iii). anantichain of P corresponds to a complete subgraph in G_P ;
(iv). contains no induced cycles of length greater than 4.

II. 3-Colored Diagram

A 3-coloured diagram Q; we consider normal edges to represent vertices in a covering relation and red edges to represent incomparable vertices or vertices in a covering relation and dashed lines to represent a chain of length three and thus constitute the3-colors and hence the name 3-*colored* diagram. The idea of 3-colored diagrams is explained as follows. Let G be a C-I graph and H be an induced subgraph of G. We note that there can be different ⊲- preserving subposets Q_i ofsome posets with G_{Q_i} isomorphic to the subgraph H. Let u, v,w be an induced pathin the direction from u to v in H. There are four possibilities in which u, v and wcan be related in the ⊲ - preserving subposets. It is possible to have u ⊲ v, u || v, v ⊲w and v || w. Each case will appear as a ⊲ - preserving subposet of four different posets. If u ⊲ v and v ⊲ w in a sub poset, then u ⊲ v ⊲w is a chain in the subposet and u, v,w is an induced path in H. If there is either u || v or v || w in a subposet Q, then there should be another chain from u to w in Q in order to have u, v, w an induced path in H. We try to capture this situation using the idea of 3-colored diagram. Suppose in ⊲ - preserving subposet Q of a poset P, there exists two elements u, v which is always connected by some chain of length three in Q. Let w be an element in Q such that either both uw and vw are red edges or any one of them is a red edge. Then in order to have a chain between u and v, there must exist an element x in Q so that u, x, v form a chain in Q. When both edges are normal, then we have the chain u, w, v in Q and hence the chain u, x, v is not required in this case. We denote the chain u, x, v by dashed lines between ux and xv in order to specify that it is possible to have the presence or absence of the chain u, x, v in Q. The presence of the chain u, x, v implies that either both of the edges uw and wv are red edges or one of them is a red edge. The absence of the chain implies that both uw and vw are normal edges in Q. We call posets having the above mentioned diagrams as 3-colored diagrams. Thus a 3-colored diagram contains normal edges, red edges and dashed lines, in which the dashed line between elements u and v will vanish, when there is a chain between u and v using normal or red edges. We can define 3-colored subposets in a similar way as discussed above. All subposets of the poset P that we consider in this paper are 3-colored diagrams. Thus by a single 3-colored diagram, we represent a collection of ⊲ - preserving subposets tobe forbidden for a poset. We sometimes use the term 3-colored subposets instead of3-colored diagrams in this paper. In a similar way the dual of a 3-colored diagram is also meaningful and represents a collection of ⊲- preserving dual subposets.

Theorem 2: (Theorem 1,[8]): Let G be a class of graphs with a forbidden induced subgraphs characterization. Let P = {P | P is a poset with $G_{T_P} \epsilon$G }.Then P has a characterization by forbidden ⊲ - preserving subposets.

Theorem 3: (Theorem 7.1.8, [7]) Let G be a graph. Then G is a line graph if and only if G contains none of the nine forbidden graphs of Figure 1 as an induced subgraph.

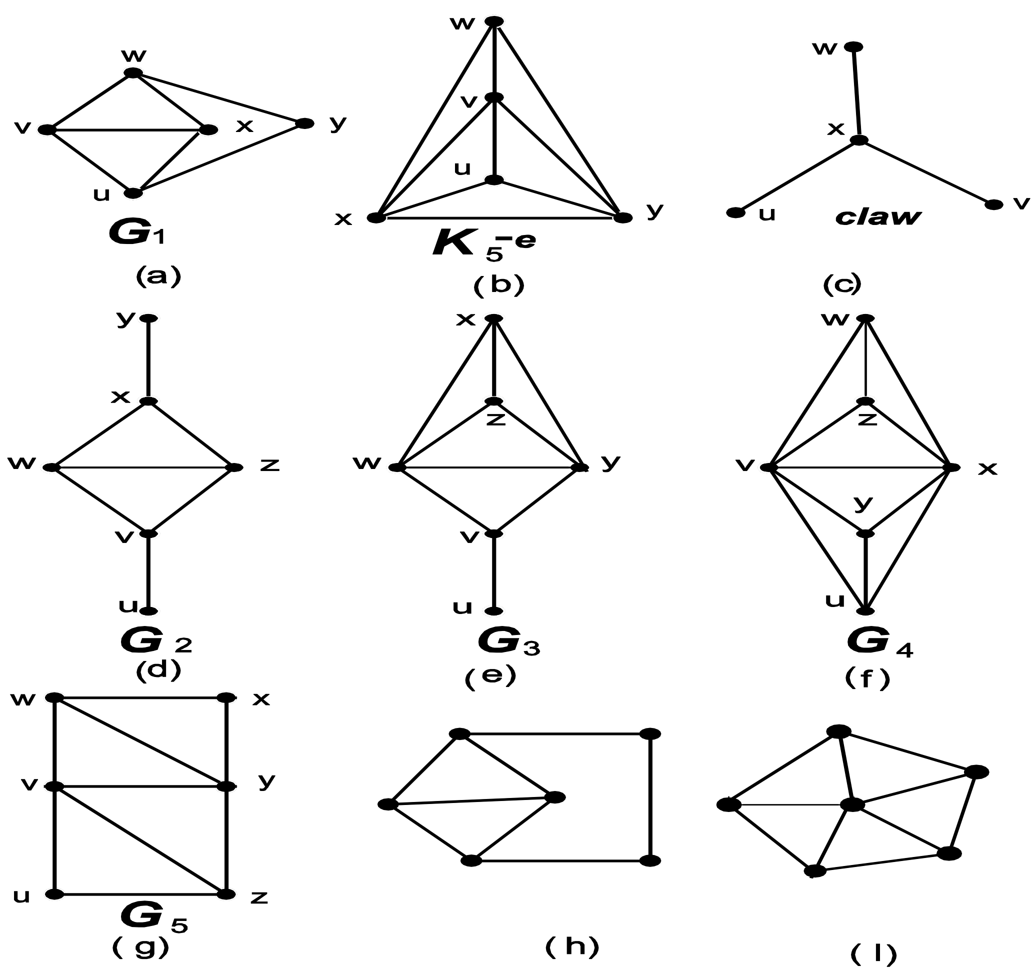


Figure 1: Nine Forbidden induced subgraphs of line graphs

Theorem 4: (Theorem 4.1,[12]) Let P be a poset. Then G_P is cograph if and only if P contains none of $T_{1,\ldots,}T_7$, depicted in Figure 2, and no duals of T_2 and T_5 as ⊲- preserving subposet.

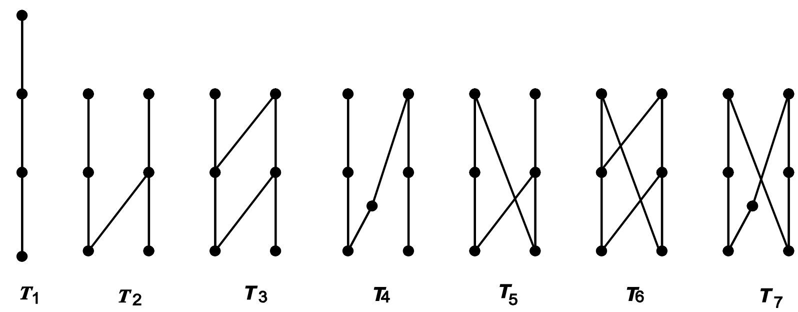


Figure 2: Forbidden $\lhd$ - preserving subposets for C-I cographs

Theorem 5: (Theorem 4,[13]) If P is a poset, then G_P is cograph if and only if P does not contain T_1 from Figure 1 and no 3-colored diagram Q_C from Figure 3 and its dual are $\lhd$ - preserving subposets .

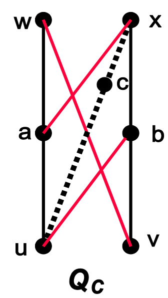


Figure 3: Forbidden $\lhd$ - preserving 3-colored subposets for C-I cographs

We consider 3-colored subposets to be forbidden so that its C-I graphs belong to the graph family F (G_3) of G_3 in Figure1

III. 3-Colored $\lhd$ - preserving subposets of posets whose c-i graphs belong to the family F (G_3)

We have the following theorem regarding the graph family F (G_3).

Theorem 6: If P is a poset, then G_P belongs to F (G_3)if and only if P contains the 3-colored diagrams Q_i; i = 2,3,4 from Figure 4 and their duals.

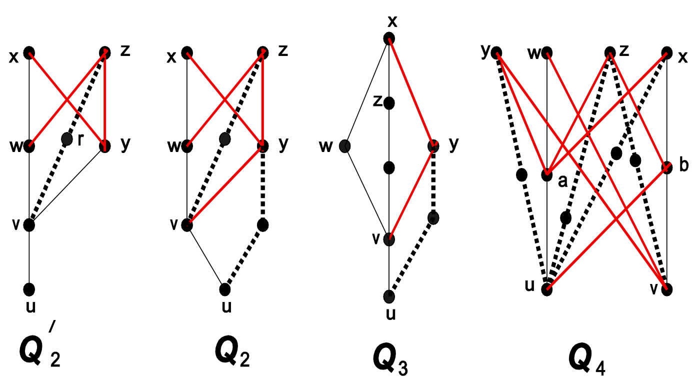


Figure 4:Forbidden 3-colored diagrams for posets whose C-I graphs contains G_3, depicted in Figure 1 (e).

Proof: Suppose P contains the 3-colored diagrams Q_i ;i = 2, 3, 4. Then clearly G_P contains the graph from Figure1 (e) as an induced subgraph.

Conversely, suppose $G_P \epsilon$F (G_3).Then G_P contains an induced subgraph G_3 shown in Figure 1(e), with vertices labeled by u, v, w, x, y and z. From the graph G_3 (Figure 1(e)), it follows that the vertex sets {u, v, w, x},{u, v, w, z}, {u, v, y, x} and {u, v, y, z} respectively induce a P_4. Since the vertices w, y, x, z induce a K_4, without loss of generality, we consider a P_4 induced by any of the above four sets, say the P_4 induced by the vertices u,v,w and x. We have already identified the $\lhd$ - preserving subposets T_i, i = 1,2

,…,7 in Theorem 4 which correspond to an induced P_4 in its C-I graph. More clearly, by Theorem 4 and Theorem 5, the chain of height 4 isomorphic to $T_1$and the 3-colored poset Q_C induce a P_4 in its corresponding C-I graph.

Now we consider two cases.

Case (1): The P_4 in G_3 induced by the vertices u, v, w and x is formed by the chain $u \lhd v \lhd w \lhd x$ in the poset P.

Case (2): The P_4 in G_3 induced by the vertices u, v, w and x is formed by two chains of length 3 as in the poset P as shown in Figure3.

We first consider Case (1).

Since w and y are adjacent in the graph G_3, either w and y are in a covering relation or these vertices are incomparable in P. The covering relation between w and y in P contradict the fact that y and v are adjacent or y and x are adjacent in G_3. Hence w||y.

Since x and z are adjacent in G_3 we have two possibilities, that is, x || z or $z \lhd x$ ($x \lhd z$ is not possible since w and z are adjacent).

Subcase (1.1): x||z.

Consider the vertex y in G_3. There are two possibilities for y with respect to v.

Either $v \lhd y$ or v|| y ($y \lhd v$ is not possible as w and y are adjacent). If $v \lhd y$ then y and u are connected by a path of length 2 follows. Now consider z. Since z is adjacent to both w and y, there are two possibilities, either $w \lhd z$ ($z \lhd w$ is not possible as z is adjacent to x) or w|| z. Similarly $y \lhd z$ or y || z.

This situation can be described into the following cases.

Subcase (1.1.1): $v \lhd y$, $w \lhd z$ and $y \lhd z$.

Subcase (1.1.2): $v \lhd y$, $w \lhd z$ and y ||z.

Subcase (1.1.3): $v \lhd y$, w || z and $y \lhd z$.

Subcase (1.1.4): $v \lhd y$, w||z and y || z.

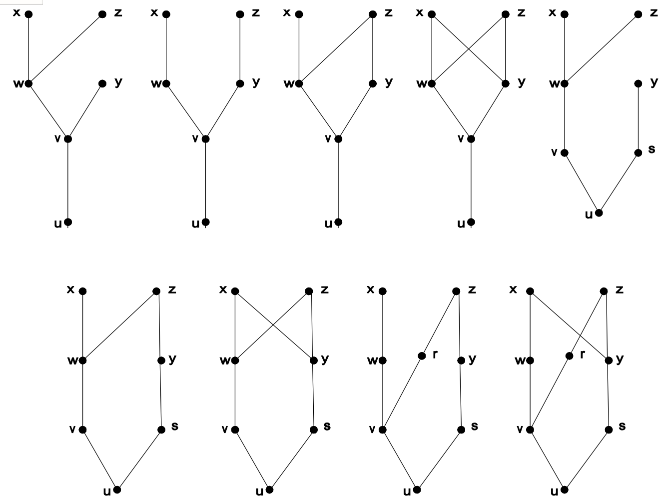
In the posets described by the subcases (1.1.1), (1.1.2) and (1.1.3), all the relations among the vertices u,v,w, x, y and z in the graph G_3 are captured and we are done. In the poset described by the subcase (1.1.4), since there is no path of length 2between v and z, we conclude that there must be a dashed line between v and z representing a chain of length 3 between v and z. Here y and x can have both possibilities, namely $y \lhd x$ or y || x and hence the edge xy can also be represented by a red edge in the poset P. Therefore we can represent the edges between w and z, y and z by red lines and the posets described in all the four subcases are captured by the 3-colored diagram Q'_2 represented in Figure 4. It is also possible that v||y. Then the poset described by the 3-colored diagram Q'_2representing all the subcases (1.1.1), (1.1.2), (1.1.3) and (1.1.4) holds if we allow a dashed line between u and y. This situation is represented in the 3-colored diagram Q_2shown in Figure 4.

Subcase (1.2): $z \lhd x$.

Since z is adjacent to w and y in G_3 and $u \lhd v \lhd w \lhd x$ in P, it follows that w || z and y || z. Since v and z are nonadjacent in G_3, there is a chain of length 3 between v and z defined by normal edges in P. Since u and y are nonadjacent in G_3, there arises two possibilities according as $v \lhd y$ (the case $y \lhd v$ is not possible as w and y are adjacent in G_3) or v || y in P.

Subcase (1.2.1): $v \lhd y$.

TheposetQ3 describes this situation without dashed lines between u and y Subcase (1.2.2): v ||y. Here, there must be a chain of length 3 between u and y in P. Both subcases are represented by the three colored diagram Q_3.Case (2): The P_4 in G_3 induced by the vertices u,v,w and x is formed by two chains of length 3 as in the poset P as shown in Figure 3.By Theorem 5, the set {u, v, w, x} will form the 3-colored diagram Q_C in Figure 3. Now we consider the vertices y and z in G_3 and find all the possibilities that these vertices can appear in the 3-colored diagram Q_C. Since there is a path of length 2 from u to y and a path of length three from u to z in G_3, there must be a chain of length 3 from u to y and u to z in P. If both these chains pass through a in Q_C, then both the vertices are in a covering relation with a ($y \not\lhd u$ and $z \not\lhd u$, since y and z are adjacent with w). Otherwise, there must be a dashed line between u and y, and u and z representing a chain of length 3 between u and y, and u and z respectively. Similar is the case between v and z in the graph G_3. Therefore, there must be a chain of length 3 from v to z in P. If the chain passes through b, then there is a covering relation between b and z ($z \not\lhd v$, since x and z are adjacent inG_3). Otherwise, there must be a dashed line between v and z representing a chain of length 3 between v and z. Since vw and vy are edges in G_3, there are two cases, either $v \lhd w$ or v || w and $v \lhd y$ or v || y and hence these edges are red. From the above discussion, analyzing all the possibilities in which the vertices y and z can be related with the 3-colored diagram Q_C, it can be verified easily that we obtain the 3-colored diagram Q_4 in Figure4 , which is an extension of Q_C. Thus we have completed all the cases in which vertices of the graph G_3 can appear in the poset P, which completes the proof of the theorem



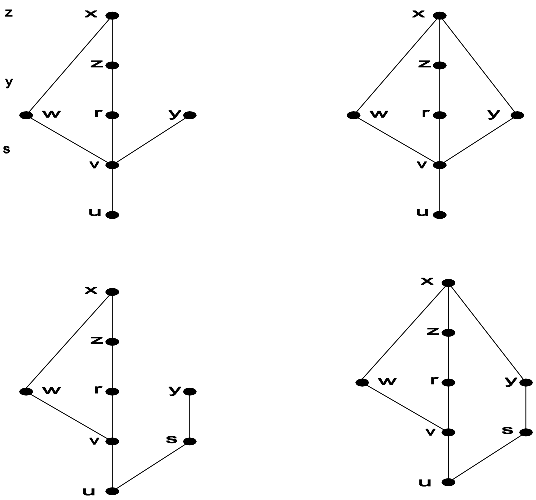
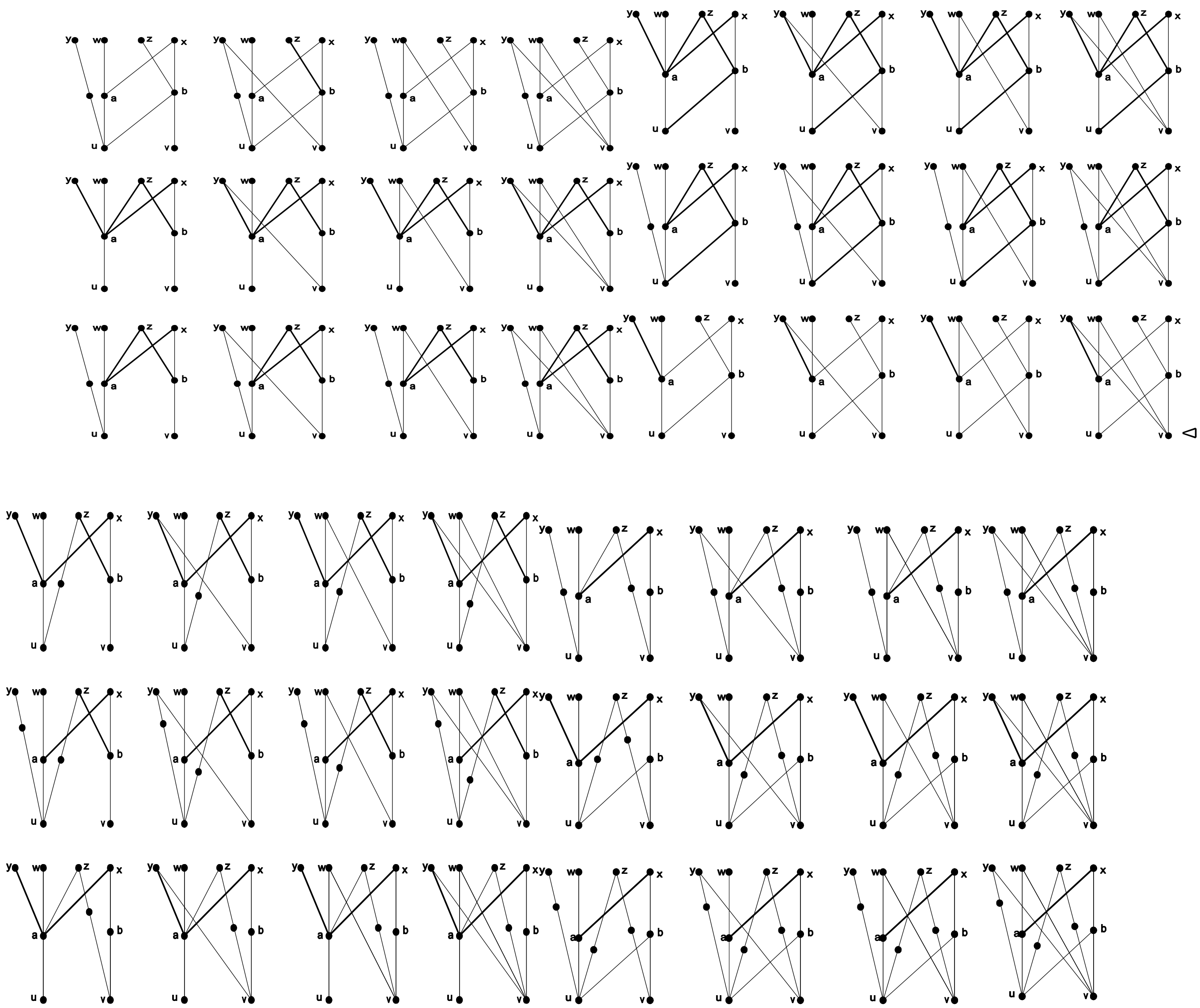


Figure 5:⊲ - preserving subposets corresponding to **Q_2** **Figure 6:**⊲ - preserving subposets corresponding to Q_3



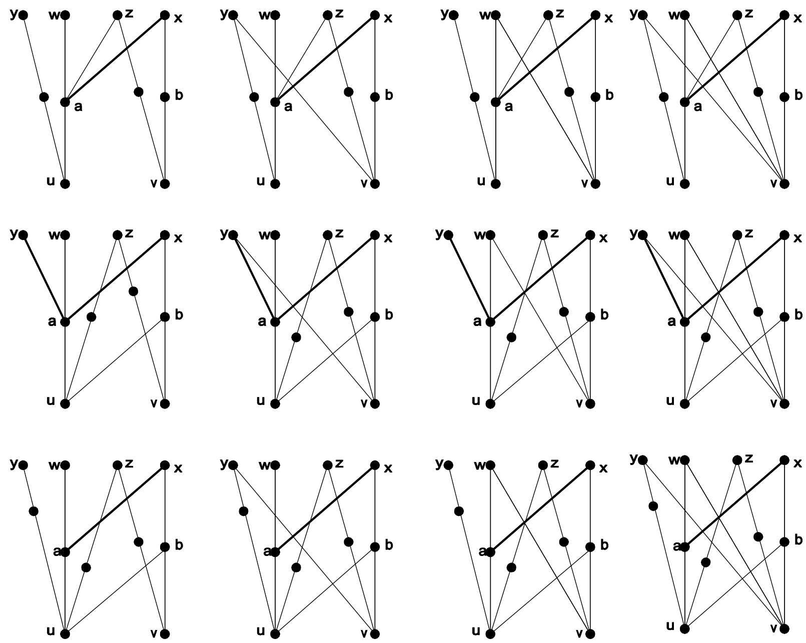


Figure 5:⊲ - preserving subposets corresponding to Q_4

II. REMARKS

The number of forbidden ⊲ - preserving subposets of a poset P is such that its C-I graph G_P belongs to a graph possessing a forbidden induced subgraph characterization as instances of the Theorem 2 is in general very large compared to the number of forbidden induced subgraphs. Here we characterize forbidden ⊲ - preserving subposets of G_3 in Figure1 and introduce the idea of 3-colored diagrams to minimize the list of subposets.

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