
$\qquad$
INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE \& ENGINEERING TECHNOLOGY
$\qquad$

# Complementary Variational Principles for the Three Fundamental Problems of Elastostatics 

Dr. C. Venkatesan ${ }^{1}$, J. Vimala ${ }^{2}$<br>${ }^{1}$ Department of Mathematics, MAHER University, Chennai.<br>${ }^{2}$ Department of Mathematics, Srinivasan College of Arts and Science, Perambalur.


#### Abstract

In this paper discusses fully the state of stress and strain in a body subjected to external forces. The Complementary Variational Principles are developed for the three fundamental problems of elastostatics and the adjoint operators are obtained. Keywords: Complementary Variational Principles, Three Fundamental Problems, Elastostatics, Displacement, Stress and Strain.


## I. INTRODUCTION

Three fundamental problems of elastostatics may be distinguished depending on what is prescribed on the surface: the first, second, and mixed problems.
The first fundamental boundary-value problem of elasticity may be stated as follows: Determine the distribution of stress and the displacements in the interior of an elastic body in equilibrium when the body forces are prescribed and the distribution of the forces acting on the surface of the body is known.
The second fundamental problem is similar, except that in this case the surface displacements rather than the forces acting on the surface are prescribed.
The third, or mixed, boundary-value problem the forces are prescribed over a portion of the surface and the displacements over the remainder. The problem of the cylinder that we are considering is of the first kind, where body forces are absent. In the following paragraphs we shall review certain basic elements of the mathematical theory of elasticity and present the mathematical formulation of the first fundamental boundary-value problem.
The equation of equilibrium of an elastic body $B_{0}$ are

$$
\begin{equation*}
\tau_{11,1}+\tau_{12,2}+\tau_{13,1}+X=0 \tag{1}
\end{equation*}
$$

$\tau_{21,1}+\tau_{22,2}+\tau_{23,3}+Y=0$
$\tau_{31,1}+\tau_{32,2}+\tau_{33,3}+Z=0$
Where $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ are the body forces per unit volume and $\tau_{11}, \tau_{22}, \ldots \ldots . . \tau_{33}$ are the components of stress tensor and $\tau_{i j, j}=\partial \tau_{i j} / \partial X_{j}$. Due to elastic symmetry the complementary shear stresses are equal $\tau_{i j}=\tau_{j i}$ for $\mathrm{i}, \mathrm{j}=1,2$ and 3
The generalized Hook's law is given by

$$
\begin{align*}
& e_{11}=a_{11} \tau_{11}+a_{12} \tau_{22}+a_{13} \tau_{33}+a_{14} \tau_{23}+a_{15} \tau_{13+} a_{16} \tau_{12} \\
& e_{22}=a_{21} \tau_{11}+a_{22} \tau_{22}+a_{23} \tau_{33}+a_{24} \tau_{23}+a_{25} \tau_{13}+a_{26} \tau_{12} \\
& e_{33}=a_{31} \tau_{11}+a_{32} \tau_{22}+a_{33} \tau_{33}+a_{34} \tau_{23}+a_{35} \tau_{13}+a_{36} \tau_{12} \\
& e_{23}=a_{41} \tau_{11}+a_{42} \tau_{22}+\cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots+a_{46} \tau_{12} \\
& e_{13}=a_{51} \tau_{11}+a_{52} \tau_{22}+\cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots+a_{56} \tau_{12} \tag{2}
\end{align*}
$$

$e_{12}=a_{61} \tau_{11}+a_{62} \tau_{22}+a_{63} \tau_{33}+a_{64} \tau_{23}+a_{65} \tau_{13}+a_{66} \tau_{12}$
$e_{11}=\partial u / \partial x, e_{22}=\partial v / \partial y, e_{33}=\partial w / \partial z$
$e_{12}=(\partial v / \partial x+\partial u / \partial y), e_{23}=(\partial v / \partial z+\partial w / \partial y) / 2$ and $e_{12}=(\partial u / \partial z+\partial w / \partial x) / 2(3)$
$\mathrm{u}, \mathrm{v}, \mathrm{w}$ being the components of the displacement vectors in the directed of $1,2,3$ or
$\mathrm{x}, \mathrm{y}, \mathrm{z}$. The equations (2) involve 21 elastic constants $\mathrm{a}_{\mathrm{ij}}(\mathrm{i}, \mathrm{j}=1,2 \ldots . .6)$.
Since $\mathrm{a}_{\mathrm{ij}}==\mathrm{a}_{\mathrm{j} i \mathrm{i}} \mathrm{i}=1,2 \ldots 6$.
Alternatively equation (2.2) may be written as

$$
\begin{aligned}
& \tau_{11}=\mathrm{A}_{11} \mathrm{e}_{11}+\mathrm{A}_{12} \mathrm{e}_{22}+\mathrm{A}_{13} \mathrm{e}_{33}+\mathrm{A}_{14} \mathrm{e}_{23}+\mathrm{A}_{15} \mathrm{e}_{13}+\mathrm{A}_{16} \mathrm{e}_{12} \\
& \tau_{22}=A_{21} e_{11}+A_{22} e_{22}+A_{23} e_{33}+ A_{24} e_{23}+A_{25} e_{13}+A_{26} e_{12} \\
& \tau_{33}=A_{31} e_{11}+A_{32} e_{22}+A_{33} e_{33}+A_{34} e_{23}+A_{35} e_{13}+A_{36} e_{12} \\
& \tau_{23}=A_{41} e_{11}+A_{42} e_{22}+A_{43} e_{33}+A_{44} e_{23}+A_{45} e_{13}+A_{46} e_{12}
\end{aligned}
$$

$$
\begin{gather*}
\tau_{13}=A_{51} e_{11}+A_{52} e_{22}+A_{53} e_{33}+A_{54} e_{23}+A_{55} e_{13}+A_{56} e_{12} \\
\tau_{12}=A_{61} e_{11}+A_{62} e_{22}+A_{63} e_{33}+A_{64} e_{23}+A_{65} e_{13}+A_{66} e_{12} \tag{4}
\end{gather*}
$$

Where $A_{i j}=A_{j i}$ for $\mathrm{i}, \mathrm{j}=1,2 \ldots . .6$ can be determined in terms of
$\mathrm{a}_{\mathrm{ij},}, \mathrm{i}, \mathrm{j}=1,2,3$, $\qquad$ . 6.
Eliminating the stress components $\tau_{11}, \tau_{22}, \ldots \ldots \tau_{12}$ from (4), (1) and using (3) we obtain the equations of equilibrium in terms of the displacements as follows.


Where the operators $A^{\prime}, B^{\prime}, \ldots$ are
$A^{\prime}=A_{11}(x x)+A_{66}(y y)+A_{55}(z z)+2 A_{56}(y z)+2 A_{15}(z x)+2 A_{16}(x y)$
$B^{\prime}=A_{66}(x x)+A_{22}(y y)+A_{44}(z z)+2 A_{24}(y z)+2 A_{46}(z x)+2 A_{26}(x y)$
$C^{\prime}=A_{55}(x x)+A_{44}(y y)+A_{33}(z z)+2 A_{34}(y z)+2 A_{35}(z x)+2 A_{45}(x y)$
$H^{\prime}=A_{16}(x x)+A_{26}(y y)+A_{45}(z z)+\left(A_{25}+A_{46}\right)(y z)+\left(A_{14}+A_{56}\right)(z x)$

$$
+\left(A_{12}+A_{66}\right)(x y)
$$

$G^{\prime}=A_{15}(x x)+A_{46}(y y)+A_{35}(z z)++\left(A_{36}+A_{45}\right)(y z)++\left(A_{13}+A_{55}\right)(z x)$
$+\left(A_{14}+A_{56}\right)(x y)$
$F^{\prime}=A_{56}(x x)+A_{24}(y y)+A_{34}(z z)++\left(A_{23}+A_{44}\right)(y z)++\left(A_{36}+A_{45}\right)(z x)$
$+\left(A_{25}+A_{46}\right)(x y)$
The symbols (xx),(yy) ,.......; are defined as
$(x x)=\partial^{2}() / \partial x^{2},(y y)=\partial^{2}() / \partial y^{2},(z z)=\partial^{2}() / \partial z^{2},(x y)=\partial^{2}() / \partial x \partial y$,
According to different boundary conditions we define the well-known fundamental problems of electrostatics as follows:

## II. FUNDAMENTAL PROBLEMS OF ELASTOSTATICS

Three fundamental problems of elastostatics may be distinguished depending on what is prescribed on the surface: the first, second, and mixed problems.

## A. First Fundamental Problem

External forces are prescribed over the whole surface; it is required to determine the stresses and displacements at any point inside the body and on the surface. Alternatively, three components of external forces may be prescribed over the surface, namely their projections on three non-coincident directions.

1) Example: The normal component $\sigma$ and two projections of tangential forces on two orthogonal directions. It is most common to coordinate axes $\mathrm{X}_{\mathrm{B}}, \mathrm{Y}_{\mathrm{B}}, \mathrm{Z}_{\mathrm{B}}$ ( n is the direction of the outward normal to the surface; the components of forces refer to unit area).
The boundary conditions are written as
i.e.

$$
\begin{gather*}
\tau_{11} l+\tau_{12} m+\tau_{13} n=X_{B} \\
\tau_{12} l+\tau_{22} m+\tau_{23} n=Y_{B} \tag{8a}
\end{gather*}
$$

$\tau_{13} l+\tau_{23} m+\tau_{33} n=Z_{B}$ on $\partial B_{0}$
Where ( $1, \mathrm{~m}, \mathrm{n}$ ) is the direction cosines of the outer normal to $\partial B_{0}$ and $X_{B}, Y_{B}, Z_{B}$ are the forces prescribed over the whole boundary $\partial B_{0}$.

## B. Second Fundamental Problem

The projections of displacement on three non-coincident directions are prescribed over the whole surface.

1) Example: The projections $U_{B}, V_{B}, W_{B}$. On the axes of a Cartesian rectangular coordinate system.

The boundary conditions are of the form
i.e. $\mathrm{u}=\mathrm{U}_{\mathrm{B}}, \quad \mathrm{v}=\mathrm{V}_{\mathrm{B}}, \mathrm{w}=\mathrm{W}_{\mathrm{B}}$ on $\partial B_{0}$

Where $\mathrm{U}_{\mathrm{B}}, \mathrm{V}_{\mathrm{B}}, \mathrm{W}_{\mathrm{B}}$ are the displacement prescribed on $\partial B_{0}$.

## C. Mixed Problem

Forces are prescribed over a part of the surface, and displacements over the remainder. The boundary conditions on the first part of the surface are written in the form of ( 8 a ), and on the second in the form of ( 8 b ). Two mixed problems also belong those where the components of the forces and two components of displacement, or one component of displacement and two components of forces, etc. depending on the shape of a body and the distribution of forces, it is often more convenient to use a suitable curvilinear coordinate system rather than a Cartesian system.
By eliminating the stress components of system symmetric equations, we obtain their equations containing only the projections of displacement. For an orthotropic body moving under the action of external forces or performing free vibrations the equations of motion in terms of the projections of displacement are of the form.
The uniqueness of the solution of the equilibrium equations for a homogenous body undergoing small stains when the strain components are linear functions of the derivatives of the displacements with respect to the co-ordinates is established by Kirchhoff's theorem.

1) Example : External forces are prescribed over $\partial B_{1}$ and the projection of displacements is prescribed over $\partial B_{2}$. Such that $\partial B_{1} U \partial B_{2}=\partial B_{0}$

## III. COMPLEMENTARY VARIATIONAL PRINCIPLES

Equation (5) along with the boundary conditions ( 8 a or b or c ) can be written in the canonical form as follows
$T U=\varnothing$
$T^{*} \emptyset=Q \quad \operatorname{in} B_{0}$
With boundary conditions (8a) or (8b) or (8c) where

$$
U=[u, v, w]^{t}
$$

$\emptyset=\left[\emptyset_{1} \emptyset_{2} \emptyset_{3} \emptyset_{4} \emptyset_{5} \emptyset_{6}\right]^{t}$
$Q=[X, Y, Z]^{t}$,
And
$T: H(U) \rightarrow H(\emptyset)$ is
$\mathrm{T}=\left[\begin{array}{llllll}T_{11} & T_{21} & T_{31} & T_{41} & T_{51} & T_{61} \\ T_{12} & T_{22} & T_{32} & T_{42} & T_{52} & T_{62} \\ T_{13} & T_{23} & T_{33} & T_{43} & T_{53} & T_{63}\end{array}\right]$
Where

$$
T_{11}=i\{D(x)+E(y)+F(z)\}+j Q_{1}(x)
$$

$T_{12}=i E(x)+j C(z), T_{13}=i F(x)+j C(y)$,
$\left.T_{21}=i G(y)+j A(z), T_{22}=i\{G(x)+H(y)+I(z)\}+j R(y)\right\}$,
$T_{23}=i I(y)+j A(x), T_{31}=j J(z)+j B(y)$,

$$
\begin{gather*}
T_{32}=i K(z)+j B(x), T_{33}=i\{(x)+K(y)+L(z)+j S(z), \\
T_{41}=i D_{1}(x)+j M(x), T_{42}=j N(y), T_{43}=j P(z), \tag{12}
\end{gather*}
$$

$T_{51}=0, T_{52}=i E_{1}(y), T_{53}=0, T_{61}=0, T_{62}=0, T_{63}=i F_{1}(z)$
Where $\mathrm{D}, \mathrm{E}, \mathrm{F}, \ldots$. Can be obtained from the value of the elastic module

$$
, i, j=1,2, \ldots .6 \text { using the following relation }
$$

$\mathrm{D}=\operatorname{Sqrt}\left(\mathrm{A}_{15} \mathrm{~A}_{16} / \mathrm{A}_{56}\right), \mathrm{E}=\operatorname{Sqrt}\left(\mathrm{A}_{16} \mathrm{~A}_{56} / \mathrm{A}_{15}\right), \mathrm{F}=\operatorname{Sqrt}\left(\mathrm{A}_{56} \mathrm{~A}_{15} / \mathrm{A}_{16}\right)$,
$\mathrm{G}=\operatorname{Sqrt}\left(\mathrm{A}_{26} \mathrm{~A}_{46} / \mathrm{A}_{24}\right), \mathrm{H}=\operatorname{Sqrt}\left(\mathrm{A}_{24} \mathrm{~A}_{26} / \mathrm{A}_{46}\right), \mathrm{I}=\operatorname{Sqrt}\left(\mathrm{A}_{24} \mathrm{~A}_{46} / \mathrm{A}_{26}\right)$,
$J=\operatorname{Sqrt}\left(\mathrm{A}_{45} \mathrm{~A}_{35} / \mathrm{A}_{34}\right), \mathrm{K}=\operatorname{Sqrt}\left(\mathrm{A}_{34} \mathrm{~A}_{45} / \mathrm{A}_{35}\right), \mathrm{L}=\operatorname{Sqrt}\left(\mathrm{A}_{34} \mathrm{~A}_{35} / \mathrm{A}_{45}\right)$,
${ }^{2}+{ }^{2}+{ }^{2}=66,{ }^{2}+{ }^{2}+{ }^{2}=55,{ }^{2}+{ }^{2}+{ }^{2}=44$,
${ }_{1}=14,=25,={ }_{36}, \mathrm{MN}=12,=13,=23$,
$1^{2}+1^{2}+{ }^{2}+{ }^{2}=11,1^{2}+{ }^{2}+{ }^{2}+{ }^{2}=22,1^{2}+{ }^{2}+{ }^{2}+{ }^{2}=33$,
$\mathrm{M}=\operatorname{Sqrt}\left(\mathrm{A}_{13} \mathrm{~A}_{12} / \mathrm{A}_{23}\right), \mathrm{N}=\operatorname{Sqrt}\left(\mathrm{A}_{12} \mathrm{~A}_{23} / \mathrm{A}_{13}\right), \mathrm{P}=\left(\mathrm{A}_{13} \mathrm{~A}_{23} / \mathrm{A}_{12}\right)$
with

$$
{ }^{*}=\quad(\quad) \rightarrow \quad(\quad) \text { and }{ }^{*}=-[\quad]
$$

Where ( ) ( )are Hilbert space with inner product ( , ) and < , >
defined as follows:
For $\Phi_{1}, \Phi_{2}, \varepsilon \Omega(\Phi)$, we get
$\left({ }_{1}, 2\right)=\int_{0}, 2$
Where $I_{1}, 2$ are matrices with vector components.
For $U_{1}, U_{2} \varepsilon \Lambda()$,
We get
$\langle 1,2\rangle=\int_{0} 1,2$,
Where ${ }_{1},{ }_{2}$ are matrices with scalar components
Thus

$$
\begin{equation*}
(\quad)=\{(,), \Omega(\emptyset)\} \text { and } \quad(\quad)=\{<,>\Lambda()\} \tag{17}
\end{equation*}
$$

are the real Hilbert spaces, In all the above equations represent the transpose of the matrix X .
The operator T and $\mathrm{T}^{*}$, satisfy the following identity
$()=,<^{*}, \quad>+() \quad$,
Where

$$
\begin{equation*}
=(\quad) \rightarrow \quad(\quad) \text { is given by } \tag{18}
\end{equation*}
$$

$\sigma=\left[\begin{array}{llllll}\sigma_{11} & \sigma_{21} & \sigma_{31} & \sigma_{41} & \sigma_{51} & \sigma_{61} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} & \sigma_{42} & \sigma_{52} & \sigma_{62} \\ \sigma_{13} & \sigma_{32} & \sigma_{33} & \sigma_{43} & \sigma_{53} & \sigma_{63}\end{array}\right]^{t}$
and

$$
\begin{align*}
& 23=+, 31=+, 32=+ \text {, } \\
& { }_{33}=\left(+\quad+\quad+\quad,{ }_{41}=1+\right.\text {, } \\
& { }_{42}=,{ }_{43}=\quad,{ }_{51}=0,{ }_{52}=1 \text {, } \\
& { }_{53}=0,{ }_{61}=0,{ }_{62}=0,{ }_{63}=1 \text {, }  \tag{20}\\
& \text { *: ( ) } \rightarrow \text { ( ) is given by } \\
& \text { * }=\text { [ ] } \tag{21}
\end{align*}
$$

Such that
( , ) ${ }_{0}=<{ }^{*}, \quad>0$
From the identities (18) and (22) we declare that $\mathrm{T}^{*}$ is the adjoint operator of T and $\mathrm{s}^{*}$ is the adjoint operator of s .
Now consider the action functional

$$
\begin{equation*}
(,)=(,)-(,) / 2-<,>-<{ }^{*},>0 \tag{23}
\end{equation*}
$$

From (18) ( , ) can be written as

$$
\begin{equation*}
I(\Phi, U)=<T^{*} \Phi, U>-(\Phi, \Phi) / 2-<Q, U>+<\sigma^{*}(\Phi-\Phi \mathrm{B}), U>\partial \mathrm{B}_{0} \tag{24}
\end{equation*}
$$

Again

$$
\begin{array}{llll}
1 & =0 & \text { implies } & \mathrm{TU}=\emptyset \\
1 & =0 & \text { implies } & \mathrm{T}^{*}=
\end{array}
$$

with

$$
<{ }^{*}(-), \quad>\quad{ }_{0}=0
$$

Now < ${ }^{*}(-), \quad>\quad 0$ is satisfied under the following conditions.
A. If U known on the whole boundary ${ }_{0}$, then

$$
=0 \text { on } \quad 0
$$

B. If the stresses are known on the whole boundary ${ }_{0}$, we get ,

* $={ }^{*}$ on 0

Since ${ }^{*}(\quad)=\quad$ where $t$ is the stress vector is given by

$$
=\left[\begin{array}{lll} 
& 1, & 2, \\
& 3
\end{array}\right]
$$

where


Hence we obtain
${ }_{1}={ }_{2}=$ and $_{3}=$ on 0
C. If the projections of displacements are known $\quad{ }_{1}$ and the stresses are known on $\quad 2$ such that $\quad 1 \quad 2=\quad 0$

Thus we obtain that the result $/=0$ and $/=0$ lead respectively to the canonical equations (9) and (10) respectively with the boundary conditions (8a) or (8b) or (8c).

The functions $\mathrm{J}(\mathrm{U})$ and $\mathrm{G}(\mathrm{F})$ are constructed in a similar way as in appendix A as follows
$\mathrm{J}(\mathrm{U})=\mathrm{I}(\mathrm{TU}, \mathrm{U})=(\mathrm{TU}, \mathrm{TU}) / 2-\langle\mathrm{Q}, \mathrm{U}\rangle$
With U satisfying the boundary condition (8a) or (8b) or (8c) and

$$
()=\{,-1()\}=-(,) / 2
$$

As the functional ( , ) is concave in $\emptyset$ and convex $U$
we get

$$
\begin{equation*}
\left({ }_{2}\right) \leq \quad(\quad)=(\quad, \quad)=(\quad) \leq(\quad 1) \tag{27}
\end{equation*}
$$

Where $\left(,{ }_{1}\right)$ and $\left(\Phi_{2}, U_{2}\right)$ are the approximate solutions and $($,$) is the exact solution of the given physical problem.$
The expression for strain energy is given by
$\left.\left.+661^{2}\right] / 2\right]$
From the definition of the functional (25) and (26) we can prove that
$-(\quad)=-\quad(\quad)$
Such that

$$
\begin{equation*}
=\int_{0} \tag{30}
\end{equation*}
$$

Thus from (27) and (29) we get

$$
\begin{equation*}
\left({ }_{1}\right) \leq \quad(\quad)=\quad=\quad(\quad) \leq \quad\left(\quad{ }_{1}\right) \tag{31}
\end{equation*}
$$

## IV. CONCLUSION

The state of stress in an elastic body may be considered knowing the components of stress, which depend on three projections of displacement on the coordinate directions. In this paper describe fully the state of stress and strain in a body subjected to external forces the complete system of equations for the determination of six components of stress and three projections of displacement are obtained by taking three equations of equilibrium. The Complementary Variational Principles are developed for the three fundamental problems of elastostatics with example and the adjoint operators are obtained.

## REFERENCES

[1] Muskhellshvilli N.I., "Some Basic Problems of the Mathematical theory of Elasticity, Torsion and Bending", Third edition, lzd. Akad. Nauk SSSR, Moscow, Leningrad, 1949 (in Russian), Transl. P. Noordhoff, Groninge.
[2] NovozhllovV.V.,"Theory of Elasticity", Sudpromiglz, Leningrad, 1958 (in Russian), Transl. Pergamon Press, New York.
[3] Timoshenko S.P. and S. Woinowsky - Krieger, "Theory of Plates and Shells", New York, McGraw Hill Book Company, Inc., (PP376).
[4] Kikuchi N., and Oden, J.T., "Contact Problems in Elasticity", Society for Industrial and applied Mathematics, Philadelphia.
[5] Fung Y.C., "Foundations of Solid Mechanics", Prentice-Hall of India, Private Limited, New Delhi .
[6] Lekhnitskil S.G., "Theory of Elasticity of an Anisotropic body", Society for Industrial and Applied Mathematics, Philadelphia.
[7] Mikhlin S.G., "Variational Methods in Mathematical Physics", Pergoman Press, Oxford
[8] Rektorys, Karel, "Variational Methods in Mathematics, Science and Engineering", D.Reidel Publishing Co., Holland .
[9] Finlayson, B.A., "The Mehod of Weighted Residuals and Variational Principles", Academic Press, New York.
[10] Kantrovich L.V. and Krylov V. I, "Approximate Methods of Higher Analysis", P. Noordhoff Ltd., The Netherlands.
[11] Atherton R.W., and Homsy G.M., "On the existence and the formulation of Variational Principles for Nonlinear Differential Equations", Stud in Appll. Maths., Vol 1 LIV No.1, Magri F., "Variational formulation for every linear equation", Int. Jl of Eng.Sci., 12, 531-549
[12] Arthurs. A. M.,"ComplementaryVariational Principles", Clarendon Press, Oxford.
[13] Borrows. B.L., "ComplementaryVariational Principles of Linear and Nonlinear Equations", J. Phy. A. Math. Gen., 14, 797-808.

do
cross ${ }^{\text {ref }}$
10.22214/IJRASET


IMPACT FACTOR: 7.129

TOGETHER WE REACH THE GOAL.

IMPACT FACTOR:
7.429

## INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE \& ENGINEERING TECHNOLOGY
Call : 08813907089 @ (24*7 Support on Whatsapp)

