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Complementary Variational Principles for the Three Fundamental Problems of Elastostatics

Dr. C. Venkatesan[1], J. Vimala[2]
[1]*Department of Mathematics, MAHER University, Chennai.*
[2]*Department of Mathematics, Srinivasan College of Arts and Science, Perambalur.*

Abstract: In this paper discusses fully the state of stress and strain in a body subjected to external forces. The Complementary Variational Principles are developed for the three fundamental problems of elastostatics and the ad joint operators are obtained.
Keywords: Complementary Variational Principles, Three Fundamental Problems, Elastostatics, Displacement, Stress and Strain.

I. INTRODUCTION

Three fundamental problems of elastostatics may be distinguished depending on what is prescribed on the surface: the first, second, and mixed problems.

The first fundamental boundary-value problem of elasticity may be stated as follows: Determine the distribution of stress and the displacements in the interior of an elastic body in equilibrium when the body forces are prescribed and the distribution of the forces acting on the surface of the body is known.

The second fundamental problem is similar, except that in this case the surface displacements rather than the forces acting on the surface are prescribed.

The third, or mixed, boundary-value problem the forces are prescribed over a portion of the surface and the displacements over the remainder. The problem of the cylinder that we are considering is of the first kind, where body forces are absent. In the following paragraphs we shall review certain basic elements of the mathematical theory of elasticity and present the mathematical formulation of the first fundamental boundary-value problem.

The equation of equilibrium of an elastic body B_0 are

$$\tau_{11,1} + \tau_{12,2} + \tau_{13,1} + X = 0$$

$$\tau_{21,1} + \tau_{22,2} + \tau_{23,3} + Y = 0$$

$$\tau_{31,1} + \tau_{32,2} + \tau_{33,3} + Z = 0 \qquad (1)$$

Where X, Y, Z are the body forces per unit volume and $\tau_{11}, \tau_{22}, \dots\dots .\tau_{33}$ are the components of stress tensor and $\tau_{ij,j} = \partial\tau_{ij} / \partial X_j$.

Due to elastic symmetry the complementary shear stresses are equal $\tau_{ij} = \tau_{ji}$ for i,j=1,2 and 3

The generalized Hook's law is given by

$$e_{11} = a_{11}\tau_{11} + a_{12}\tau_{22} + a_{13}\tau_{33} + a_{14}\tau_{23} + a_{15}\tau_{13+}a_{16}\tau_{12}$$

$$e_{22} = a_{21}\tau_{11} + a_{22}\tau_{22} + a_{23}\tau_{33} + a_{24}\tau_{23} + a_{25}\tau_{13} + a_{26}\tau_{12}$$

$$e_{33} = a_{31}\tau_{11} + a_{32}\tau_{22} + a_{33}\tau_{33} + a_{34}\tau_{23} + a_{35}\tau_{13} + a_{36}\tau_{12}$$

$$e_{23} = a_{41}\tau_{11} + a_{42}\tau_{22} + \cdots \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . + a_{46}\tau_{12}$$

$$e_{13} = a_{51}\tau_{11} + a_{52}\tau_{22} + \cdots \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . + a_{56}\tau_{12}$$

$$e_{12} = a_{61}\tau_{11} + a_{62}\tau_{22} + a_{63}\tau_{33} + a_{64}\tau_{23} + a_{65}\tau_{13} + a_{66}\tau_{12} \qquad (2)$$

$$e_{11} = \partial u/\partial x, e_{22} = \partial v/\partial y, e_{33} = \partial w/\partial z$$

$$e_{12} = (\partial v/\partial x + \partial u/\partial y), e_{23} = (\partial v/\partial z + \partial w/\partial y)/2 \text{ and} e_{12} = (\partial u/\partial z + \partial w/\partial x)/2 (3)$$

u,v,w being the components of the displacement vectors in the directed of 1,2,3 or x,y,z. The equations (2) involve 21 elastic constants ,a_{ij} (i,j=1,2….6).

Since a_{ij}==a_{ji}i,j=1,2…6.

Alternatively equation (2.2) may be written as

$$\tau_{11} = A_{11}e_{11} + A_{12}e_{22} + A_{13}e_{33} + A_{14}e_{23} + A_{15}e_{13} + A_{16}e_{12}$$

$$\tau_{22} = A_{21}e_{11} + A_{22}e_{22} + A_{23}e_{33} + A_{24}e_{23} + A_{25}e_{13} + A_{26}e_{12}$$

$$\tau_{33} = A_{31}e_{11} + A_{32}e_{22} + A_{33}e_{33} + A_{34}e_{23} + A_{35}e_{13} + A_{36}e_{12}$$

$$\tau_{23} = A_{41}e_{11} + A_{42}e_{22} + A_{43}e_{33} + A_{44}e_{23} + A_{45}e_{13} + A_{46}e_{12}$$

$$\tau_{13} = A_{51}e_{11} + A_{52}e_{22} + A_{53}e_{33} + A_{54}e_{23} + A_{55}e_{13} + A_{56}e_{12}$$

$$\tau_{12} = A_{61}e_{11} + A_{62}e_{22} + A_{63}e_{33} + A_{64}e_{23} + A_{65}e_{13} + A_{66}e_{12} \quad (4)$$

Where $A_{ij} = A_{ji}$ for i,j =1,2…. 6 can be determined in terms of a_{ij},i,j =1,2,3,…...…6.

Eliminating the stress components $\tau_{11}, \tau_{22}, \ldots \ldots \tau_{12}$ from (4), (1) and using (3) we obtain the equations of equilibrium in terms of the displacements as follows.

$$\begin{bmatrix} A' & H' & G' \\ H' & B' & F' \\ G' & F' & C' \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (5)$$

Where the operators A', B', ….are

$$A' = A_{11}(xx) + A_{66}(yy) + A_{55}(zz) + 2A_{56}(yz) + 2A_{15}(zx) + 2A_{16}(xy)$$

$$B' = A_{66}(xx) + A_{22}(yy) + A_{44}(zz) + 2A_{24}(yz) + 2A_{46}(zx) + 2A_{26}(xy)$$

$$C' = A_{55}(xx) + A_{44}(yy) + A_{33}(zz) + 2A_{34}(yz) + 2A_{35}(zx) + 2A_{45}(xy)$$

$$H' = A_{16}(xx) + A_{26}(yy) + A_{45}(zz) + (A_{25} + A_{46})(yz) + (A_{14} + A_{56})(zx) + (A_{12} + A_{66})(xy)$$

$$G' = A_{15}(xx) + A_{46}(yy) + A_{35}(zz) + +(A_{36} + A_{45})(yz) + +(A_{13} + A_{55})(zx) + (A_{14} + A_{56})(xy)$$

$$F' = A_{56}(xx) + A_{24}(yy) + A_{34}(zz) + +(A_{23} + A_{44})(yz) + +(A_{36} + A_{45})(zx) + (A_{25} + A_{46})(xy) \quad (6)$$

The symbols (xx),(yy) ,……..; are defined as

$$(xx) = \partial^2(\,)/\partial x^2, (yy) = \partial^2(\,)/\partial y^2, (zz) = \partial^2(\,)/\partial z^2, (xy) = \partial^2(\,)/\partial x \partial y, \quad (7)$$

According to different boundary conditions we define the well-known fundamental problems of electrostatics as follows:

II. FUNDAMENTAL PROBLEMS OF ELASTOSTATICS

Three fundamental problems of elastostatics may be distinguished depending on what is prescribed on the surface: the first, second, and mixed problems.

A. *First Fundamental Problem*

External forces are prescribed over the whole surface; it is required to determine the stresses and displacements at any point inside the body and on the surface. Alternatively, three components of external forces may be prescribed over the surface, namely their projections on three non-coincident directions.

1) *Example:* The normal component σ and two projections of tangential forces on two orthogonal directions. It is most common to coordinate axes X_B, Y_B, Z_B (n is the direction of the outward normal to the surface; the components of forces refer to unit area).

The boundary conditions are written as

i.e.

$$\tau_{11}l + \tau_{12}m + \tau_{13}n = X_B$$

$$\tau_{12}l + \tau_{22}m + \tau_{23}n = Y_B$$

$$\tau_{13}l + \tau_{23}m + \tau_{33}n = Z_B \text{ on } \partial B_0 \quad (8a)$$

Where (l, m, n) is the direction cosines of the outer normal to ∂B_0 and X_B, Y_B, Z_B are the forces prescribed over the whole boundary ∂B_0.

B. *Second Fundamental Problem*

The projections of displacement on three non-coincident directions are prescribed over the whole surface.

1) *Example:* The projections U_B, V_B, W_B . On the axes of a Cartesian rectangular coordinate system.

The boundary conditions are of the form

i.e. $u = U_B$, $v = V_B$, $w = W_B$ on ∂B_0 (8b)

Where U_B, V_B, W_B are the displacement prescribed on ∂B_0.

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C. Mixed Problem

Forces are prescribed over a part of the surface, and displacements over the remainder. The boundary conditions on the first part of the surface are written in the form of (8a), and on the second in the form of (8b). Two mixed problems also belong those where the components of the forces and two components of displacement, or one component of displacement and two components of forces, etc. depending on the shape of a body and the distribution of forces, it is often more convenient to use a suitable curvilinear coordinate system rather than a Cartesian system.

By eliminating the stress components of system symmetric equations, we obtain their equations containing only the projections of displacement. For an orthotropic body moving under the action of external forces or performing free vibrations the equations of motion in terms of the projections of displacement are of the form.

The uniqueness of the solution of the equilibrium equations for a homogenous body undergoing small stains when the strain components are linear functions of the derivatives of the displacements with respect to the co-ordinates is established by Kirchhoff's theorem.

1) *Example :* External forces are prescribed over ∂B_1 and the projection of displacements is prescribed over∂B_2.Such that$\partial B_1 \, U \, \partial B_2 = \partial B_0$ (8c)

III. COMPLEMENTARY VARIATIONAL PRINCIPLES

Equation (5) along with the boundary conditions (8 a or b or c) can be written in the canonical form as follows

$TU = \emptyset$ (9)

$T^{*}\emptyset = Q$ inB_0 (10)

With boundary conditions (8a) or (8b) or (8c) where

$$U = [u, v, w]^t$$

$\emptyset = [\emptyset_1 \emptyset_2 \emptyset_3 \emptyset_4 \emptyset_5 \emptyset_6\,]^t$

$Q = [X, Y, Z]^t$,

And

$T: H(U) \rightarrow H(\emptyset)$is

$$T = \begin{bmatrix} T_{11} & T_{21} & T_{31} & T_{41} & T_{51} & T_{61} \\ T_{12} & T_{22} & T_{32} & T_{42} & T_{52} & T_{62} \\ T_{13} & T_{23} & T_{33} & T_{43} & T_{53} & T_{63} \end{bmatrix} \quad (11)$$

Where

$$T_{11} = i\{D(x) + E(y) + F(z)\} + jQ_1(x),$$

$T_{12} = iE(x) + jC(z)$,$T_{13} = iF(x) + jC(y)$,

$T_{21} = iG(y) + jA(z), T_{22} = i\{G(x) + H(y) + I\,(z)\} + jR(y)\}$,

$T_{23} = iI(y) + jA(x), T_{31} = jJ(z) + jB(y),$

$$T_{32} = iK(z) + jB(x), T_{33} = i\{J(x) + K(y) + L(z) + jS(z),$$

$$T_{41} = iD_1(x) + jM(x), T_{42} = jN(y), T_{43} = jP(z),$$

$T_{51} = 0, T_{52} = iE_1(y), T_{53} = 0$,$T_{61} = 0$,$T_{62} = 0, T_{63} = iF_1(z)$ (12)

Where D, E, F,…. Can be obtained from the value of the elastic module

, i,j =1,2,….6 using the following relation

D = Sqrt($A_{15}A_{16}/A_{56}$),E= Sqrt($A_{16}A_{56}/A_{15}$),F= Sqrt($A_{56}A_{15}/A_{16}$),

G = Sqrt($A_{26}A_{46}/A_{24}$),H= Sqrt($A_{24}A_{26}/A_{46}$), I = Sqrt($A_{24}A_{46}/A_{26}$),

J= Sqrt($A_{45}A_{35}/A_{34}$), K = Sqrt($A_{34}A_{45}/A_{35}$), L = Sqrt($A_{34}A_{35}/A_{45}$),

$^2 + \; ^2 + \; ^2 = \; _{66}, \; ^2 + \; ^2 + \; ^2 = \; _{55}, \; ^2 + \; ^2 + \; ^2 = \; _{44},$

$_1 \; = \; _{14}, \; = \; _{25}, \; = \; _{36}$,MN= $_{12}, \; = \; _{13}, \; = \; _{23},$

$_1^2 + \; _1^2 + \; ^2 + \; ^2 = \; _{11}, \; _1^2 + \; ^2 + \; ^2 + \; ^2 = \; _{22}, \; _1^2 + \; ^2 + \; ^2 + \; ^2 = \; _{33},$

M = Sqrt($A_{13}A_{12}/A_{23}$), N = Sqrt($A_{12}A_{23}/A_{13}$),P= ($A_{13}A_{23}/A_{12}$) (13)

with

$^{*} = \; (\;) \rightarrow \; (\;)$and $^{*} = -[\;]$, (14)

Where ( ) ( )are Hilbert space with inner product (,) and < , >

defined as follows:

For $\Phi_1, \Phi_2, \varepsilon\, \Omega\,(\Phi)$, we get

$(\ _1, \ _2) = \int_0 \ \ , \ _2$ (15)

Where $_1, \ _2$ are matrices with vector components.

For $U_1, U_2\ \varepsilon\ \Lambda(\ )$,

We get

$< \ _1, \ _2 >= \int_0 \ _1 \ , \ _2 \ ,$ (16)

Where $_1, \ _2$ are matrices with scalar components

Thus

$(\ ) = \{(\ ,\ ), \Omega(\emptyset)\}$ and $(\ ) = \{<\ ,\ > \Lambda(\ )\}$ (17)

are the real Hilbert spaces, In all the above equations represent the transpose of the matrix X .

The operator T and T*, satisfy the following identity

$(\ ,\ ) = <\ ^*\ ,\ > + (\ ,\ )\ _0$ (18)

Where

$= (\ ) \rightarrow (\ )$ is given by

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} & \sigma_{41} & \sigma_{51} & \sigma_{61} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} & \sigma_{42} & \sigma_{52} & \sigma_{62} \\ \sigma_{13} & \sigma_{32} & \sigma_{33} & \sigma_{43} & \sigma_{53} & \sigma_{63} \end{bmatrix}^t \quad (19)$$

and

$_{11} = (\ + \ + \ ) + \ _1\ ,\ _{12} = \ + \ ,$

$_{13} = \ + \ ,\ _{21} = \ + \ ,\ _{22} = (\ + \ + \ ) + \ ,$

$_{23} = \ + \ ,\ _{31} = \ + \ ,\ _{32} = \ + \ ,$

$_{33} = (\ + \ + \ ) + \ ,\ _{41} = \ _1 + \ ,$

$_{42} = \ ,\ _{43} = \ ,\ _{51} = 0,\ _{52} = \ _1\ ,$

$_{53} = 0,\ _{61} = 0,\ _{62} = 0,\ _{63} = \ _1\ ,$ (20)

$^*: (\ ) \rightarrow (\ )$ is given by

$^* = [\ ]$ (21)

Such that

$(\ ,\ )\ _0 = <\ ^*\ ,\ >\ _0$ (22)

From the identities (18) and (22) we declare that T* is the adjoint operator of T and s* is the adjoint operator of s.

Now consider the action functional

$(\ ,\ ) = (\ ,\ ) - (\ ,\ )/2 - <\ ,\ > - <\ ^*\ ,\ >\ _0$ (23)

From (18) $(\ ,\ )$ can be written as

$$I(\Phi, U) = < T^*\Phi, U > -(\Phi, \Phi)/2 - < Q, U > + < \sigma^*(\Phi - \Phi B), U > \partial B_0 \quad (24)$$

Again

$/ \ = 0$ implies TU = $\emptyset$

$/ \ = 0$ implies T* =

with

$<\ ^*(\ -\ ),\ >\ _0 = 0$

Now $<\ ^*(\ -\ ),\ >\ _0 = 0$ is satisfied under the following conditions.

A. If U known on the whole boundary $_0$, then

$= 0$ on $_0$

B. If the stresses are known on the whole boundary $_0$, we get ,

$^* = ^*$ on $_0$

Since $^*(\ ) =$ where t is the stress vector is given by

$$= [\ _1, \ _2, \ _3]$$

where

$_1 = \ _{11} + \ _{12} + \ _{13}$, $_2 = \ _{12} + \ _{22} + \ _{23}$, $_3 = \ _{13} + \ _{23} + \ _{33}$

and

$$^* = [\qquad]$$

Hence we obtain

$_1 = \quad _2 =$ and $_3 =$ on $_0$

C. If the projections of displacements are known $_1$ and the stresses are known on $_2$ such that $_1 \quad _2 = \quad _0$

Thus we obtain that the result $/ \ = 0$ and $/ \ = 0$ lead respectively to the canonical equations (9) and (10) respectively with the boundary conditions (8a) or (8b) or (8c).

The functions J(U) and G(F) are constructed in a similar way as in appendix A as follows

J (U) =I (TU, U) = (TU, TU)/2-< Q, U> (25)

With U satisfying the boundary condition (8a) or (8b) or (8c) and

$(\ ) = \ \{ \ , \ ^{-1}(\ )\} = -(\ , \ )/2$ (26)

As the functional $(\ , \ )$ is concave in ∅ and convex U

we get

$(\ _2) \leq \ (\ ) = \ (\ , \ ) = \ (\ ) \leq \ (\ _1)$ (27)

Where$(\ , \ _1)$ and (Φ_2, U_2) are the approximate solutions and $(\ , \ )$ is the exact solution of the given physical problem.

The expression for strain energy is given by

$$= [\ _{11} \ _{11}^2 + 2 \ _{12} \ _{11} \ _{22} + 2 \ _{13} \ _{11} \ _{13} + +2 \ _{14} \ _{11} \ _{23}$$
$$+2 \ _{15} \ _{11} \ _{13} + 2 \ _{16} \ _{11} \ _{12} + \ _{22} \ _{22}^2 + 2 \ _{23} \ _{22} \ _{33}$$
$$+2 \ _{24} \ _{22} \ _{23} + 2 \ _{25} \ _{22} \ _{13} + 2 \ _{26} \ _{22} \ _{12} + \ _{33} \ _{33}^2$$
$$+2 \ _{34} \ _{33} \ _{23} + 2 \ _{35} \ _{33} \ _{13} + 2 \ _{36} \ _{33} \ _{12} + \ _{44} \ _{23}^2$$
$$+2 \ _{45} \ _{23} \ _{13} + 2 \ _{46} \ _{23} \ _{12} + \ _{55} \ _{13}^2 + 2 \ _{56} \ _{13} \ _{12}$$
$$+ \ _{66} \ _{12}^2]/2] \quad (28)$$

From the definition of the functional (25) and (26) we can prove that

$- \ (\ ) = \quad = - \ (\ )$ (29)

Such that

$= \int_0$ (30)

Thus from (27) and (29) we get

$(\ _1) \leq \ (\ ) = \quad = \ (\ ) \leq \ (\ _1)$ (31)

IV. CONCLUSION

The state of stress in an elastic body may be considered knowing the components of stress, which depend on three projections of displacement on the coordinate directions. In this paper describe fully the state of stress and strain in a body subjected to external forces the complete system of equations for the determination of six components of stress and three projections of displacement are obtained by taking three equations of equilibrium. The Complementary Variational Principles are developed for the three fundamental problems of elastostatics with example and the adjoint operators are obtained.

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