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Fixed Point Theorem and Consequences in D*-Metric Spaces

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Abstract: The purpose of this paper is to establish a fixed point theorem for quasi-contractions on D*-metric spaces and obtain certain consequences

Keywords: Quasi-contraction, generalized quasi-contraction, orbit of x under f of length n

I. INTRODUCTION

The purpose of this paper is to establish a fixed point theorem for quasi-contractions on D^* -metric spaces and obtain certain consequences. In fact, we prove that the fixed point theorem for quasi-contractions on metric spaces, proved by Lj. B. Ciric [1] as a particular case of the main result of this paper. The notion of Quasi-contraction defined for selfmaps of metric spaces given by Lj. B. Ciric [2] has been extended to the selfmaps of D^* -metric spaces. The notion of quasi-contractions has been extended to include a wider class of selfmaps of metric spaces by Fisher [3] and we distinguish them as generalized quasi-contractions. Here we define below generalized quasi-contractions among selfmaps of D^* -metric spaces (X, D^*).

II. PRELIMINARIES

A. Definition: A selfmap f of a D*-metric space (X, D^*) is called a Quasi-contraction, if there is a number q with $0 \le q < 1$ such that

$$D^*(fx, fy, fy) \le q.\max\{D^*(x, y, y), D^*(x, fx, fx), D^*(y, fy, fy), \}$$

$$D^*(x, fy, fy), D^*(y, fx, fx) \}$$

for all $x, y \in X$.

B. Definition: A selfmap f of a D*-metric space (X, D^*) is called a *generalized quasi-contraction*, if for some fixed positive integers k and l, there is a number q with $0 \le q < 1$ such that

$$D^*(f^kx, f^ly, f^ly) \le q \max \left\{ D^*(f^rx, f^sy, f^sy), D^*(f^rx, f^{r'}x, f^{r'}x), \\ D^*(f^sy, f^{s'}y, f^{s'}y) : 0 \le r, r' \le k; 0 \le s, s' \le l \right\}$$

for all $x, y \in X$.

C. Definition: Let f be a selfmap of a D^* -metric space (X, D^*) and $x \in X$, $n \ge 1$ be an integer. The orbit of x under f of length n, denoted by $O_f(x:n)$, is defined by

$$O_f(x:n) = \left\{x, fx, f^2x, \ldots, f^nx\right\}$$

We define the diameter $\delta(A)$ of a set A in a D^* -metric space (X, D^*) by $\delta(A) = \frac{Sup}{x, y \in A} \{D^*(x, y, y)\}$

The following Lemmas are use full in proving fixed point theorems of quasi-contractions on D^* -metric spaces:



Suppose f is a quasi-contraction with constant q on a D^* -metric space (X, D^*) and n be a positive integer. Then for each $x \in X$ and all integers $i, j \in \{1, 2, 3, ..., n\}$,

$$D^*(f^i x, f^j x, f^j x) \leq q.\delta \left[O_f(x:n)\right] < \delta \left[O_f(x:n)\right].$$

Let $x \in X$ be arbitrary, $n \ge 1$ be an integer and $i, j \in \{1, 2, 3, ..., n\}$. Then $f^{i-1}x, f^{j-1}x, f^{i}x, f^{j}x \in O_f(x;n)$ and since f is a quasi-contraction,

$$D^{*}(f^{i}x, f^{j}x, f^{j}x) = D^{*}(ff^{i-1}x, ff^{j-1}x, ff^{j-1}x)$$

$$\leq q.\max\{D^{*}(f^{i-1}x, f^{j-1}x, f^{j-1}x), D^{*}(f^{i-1}x, f^{i}x, f^{i}x), D^{*}(f^{i-1}x, f^{j}x, f^{j}x), D^{*}(f^{j-1}x, f^{j}x, f^{j}x)\}$$

$$\leq q.Sup\{D^{*}(u, v, v,) : u, v \in O_{f}(x: n)\}$$

$$= q.\delta[O_{f}(x: n)]$$

$$<\delta \left[O_{f}\left(x:n\right)\right]$$

D. Lemma: Suppose f is a quasi-contraction with constant q on a D*-metric space (X, D^*) and $x \in X$, then for every positive integer n, there exists positive integer $k \le n$, such that

$$D^*(x, f^k x, f^k x) = \delta \left[O_f(x:n) \right]$$

1) *Proof:* If possible assume that the result is not true. This implies that there is positive integer *m* such that for all $k \le m$, we have $D^*(x, f^k x, f^k x) \ne \delta [O_f(x;m)]$. Since $O_f(x;m)$ contains *x* and $f^k x$ for $k \le m$, it follows that

$$D^*(x, f^*x, f^*x) < \delta \Big[O_f(x:m) \Big]$$

Since $O_f(x:m)$ is closed, there exists $i, j \in \{1, 2, 3, ..., m\}$ such that $D^*(x, f^kx, f^kx) = \delta [O_f(x:m)]$, contradicting the Lemma 3.2.1. Therefore

$$D^*(x, f^k x, f^k x) = \delta \left[O_f(x:n) \right]$$
 for some $k \le n$.

E. Lemma: Suppose f is a quasi-contraction with constant q on a D^* -metric space (X, D^*) , then

$$\delta \Big[O_f (x : \infty) \Big] \leq \frac{1}{1-q} D^* (x, fx, fx) \text{ for all } x \in X.$$

1) Proof: Let $x \in X$ be arbitrary. Since $O_f(x:1) \subseteq O_f(x:2) \subseteq \ldots \subseteq O_f(x:n) \subseteq O_f(x:n+1) \subseteq \ldots$, we get that



$$\delta \Big[O_f(x:1) \Big] \le \delta \Big[O_f(x:2) \Big] \le \ldots \le \delta \Big[O_f(x:n) \Big] \le \delta \Big[O_f(x:n+1) \Big] \le \ldots, \text{ showing}$$

$$\lim_{n \to \infty} \delta \Big[O_f(x:n) \Big] = Sup \Big\{ \delta \Big[O_f(x:n) \Big] : n = 1, 2, 3, \ldots \Big\}.$$

III. MAIN RESULT

A. Theorem: Suppose f is a quasi-contraction with constant q on a D^* -metric space (X, D^*) and X is f-orbitally complete. Then f has a unique fixed point $u \in X$. In fact,

B.
$$u = \lim_{n \to \infty} f^n x$$
 for any $x \in X$

and

C.
$$D^*(f^n x, u, u) \leq \frac{q^n}{1-q} D^*(x, fx, fx)$$
 for all $x \in X$, $n \geq 1$.

1) Proof: Let x be an arbitrary point of X. We claim that $\{f^n x\}$ is a Cauchy sequence in X. Let m, n be any positive integers with n < m. Since f is quasi-contraction,

$$D^*(f^n x, f^m x, f^m x) = D^*(ff^{n-1}x, ff^{m-1}x, ff^{m-1}x)$$
$$\leq q.\delta \Big[O_f(f^{n-1}x:m-n+1)\Big]$$

That is,

$$D^* \Big(f^n x, f^m x, f^m x \Big) \le q \cdot \delta \Big[O_f \Big(f^{n-1} x : m-n+1 \Big) \Big]$$

According to the Lemma 2.3, there exists an integer k_1 , with $0 \le k_1 \le m - n + 1$, such that

$$\delta \left[O_f \left(f^{n-1} x : m-n+1 \right) \right] = D * \left(f^{n-1} x, f^{k_1} f^{n-1} x, f^{k_1} f^{n-1} x \right)$$

E.

Using Lemma 2.4, we get

$$D * (f^{n-1}x, f^{k_1}f^{n-1}x, f^{k_1}f^{n-1}x) = D * (ff^{n-2}x, f^{k_1+1}f^{n-2}x, f^{k_1+1}f^{n-2}x)$$

$$\leq q.\delta \Big[O_f (f^{n-2}x:k_1+1) \Big]$$

$$\leq q.\delta \Big[O_f (f^{n-2}x:m-n+2) \Big]$$

(Since $k_1 + 1 \le m - n + 2$)

Thus



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$$D^* \Big(f^{n-1}x, f^{k_1} f^{n-1}x, f^{k_1} f^{n-1}x \Big) \le q \cdot \delta \Big[O_f \Big(f^{n-2}x : m-n+2 \Big) \Big]$$
F.

From (3.4), (3.5) and (3.6) we get

$$D^*(f^n x, f^m x, f^m x) \le q \cdot \delta \left[O_f(f^{n-1}x : m-n+1) \right]$$
$$\le q^2 \cdot \delta \left[O_f(f^{n-2}x : m-n+2) \right]$$

Proceeding in this manner, we obtain

$$D^*(f^nx, f^mx, f^mx) \leq q^n \cdot \delta \Big[O_f(x:m)\Big]$$

Using Lemma 2.3, we get

$$D^*\left(f^n x, f^m x, f^m x\right) \leq \frac{q^n}{1-q} \mathcal{S}\left[O_f\left(x:m\right)\right]$$

G.

Letting $n \to \infty$ and since $\frac{\lim_{n \to \infty} q^n = 0}{n \to \infty}$, we get that $\{f^n x\}$ is Cauchy sequence. Again X being f-orbitally complete and $\{f^n x\}$ is a Cauchy sequence in $O_f(x : \infty)$, there is a point $u \in X$ such that $u = \frac{\lim_{n \to \infty} f^n x}{n \to \infty}$.

We shall now show that u is a fixed point of f.

Consider,

$$D^{*}(u, fu, fu) \leq D^{*}(u, f^{n+1}u, f^{n+1}u) + D^{*}(f^{n+1}u, fu, fu)$$
$$= D^{*}(u, f^{n+1}u, f^{n+1}u) + D^{*}(ff^{n}u, fu, fu)$$

$$\leq D^{*}(u, f^{n+1}u, f^{n+1}u) + q. \max \{ D^{*}(f^{n}u, u, u), \\ D^{*}(f^{n}u, f^{n+1}u, f^{n+1}u), D^{*}(u, fu, fu), \\ D^{*}(f^{n}u, fu, fu), D^{*}(u, f^{n+1}u, f^{n+1}u) \}$$



$$\leq D^{*}(u, f^{n+1}u, f^{n+1}u) + q.\left\{D^{*}(f^{n}u, f^{n+1}u, f^{n+1}u) + D^{*}(f^{n}u, u, u) + D^{*}(u, fu, fu) + D^{*}(f^{n+1}u, u, u)\right\}$$

Letting $n \to \infty$ and since $\lim_{n \to \infty} f^n x = u$, we get

 $D^*(u, fu, fu) = 0$ and hence fu = u, showing that u is fixed point of f.

To prove the uniqueness, let u, u' be two fixed points of f. That is, fu = u, fu' = u'

$$D^{*}(u, u', u') = D^{*}(fu, fu', fu')$$

$$\leq q. \max \{ D^{*}(u, u', u'), D^{*}(u, fu, fu), D^{*}(u', fu', fu'),$$

$$D^{*}(u, fu', fu'), D^{*}(u', fu, fu) \}$$

$$D^{*}(u, u', u') \leq q. \max \{ D^{*}(u, u', u'), D^{*}(u, fu, fu), D^{*}(u', fu', fu'),$$

$$D^{*}(u, fu', fu'), D^{*}(u', fu, fu) \}$$

That is, $D^*(u, u', u') \le q.D^*(u, u', u')$

Since q < 1, $D^*(u, u', u') = 0$, which implies that u = u'.

Letting $n \to \infty$ in (3.7) we get (3.3). This completes the proof of the theorem.

H. Theorem: Suppose f is a selfmap of a D^* -metric space (X, D^*) and X is f-orbitally complete. If there is a positive integer k such that f^k is a quasi-contraction with constant q. Then f has a unique fixed point $u \in X$. In fact,

$$u = \lim_{n \to \infty} f^n x$$
for any $x \in X$

and

$$D^*(f^n x, u, u) \leq \frac{q^n}{1-q} a(x)$$
for all $x \in X$, $n \geq 1$,

where $a(x) = \max\left\{D^*(f^i x, f^{i+k} x, f^{i+k} x): i = 1, 2, 3, ...\right\}$ and $m = \left[\frac{n}{k}\right]$, the greatest integer not exceeding $\frac{n}{k}$.



1) Proof: Suppose f^k is a quasi-contraction of a D^* -metric space (X, D^*) . It has unique fixed point by Theorem 3.1. Let u be a fixed point of f^k . Then we claim that fu is also a fixed point of f^k . In fact,

$$f^{k}(fu) = f^{k+1}u = f(f^{k}u) = fu$$

By the uniqueness of fixed point of f^k , it follows that fu = u, showing that u is a fixed point of f. Uniqueness of the fixed point of f can be proved as in the Theorem 3.3.1.

To prove (3.10), let *n* be any integer. Then by the division algorithm, we have, n = mk + j, $0 \le j < k$, $m \ge 0$

Therefore $x \in X$, $f^n x = (f^k)^m f^j x$, since f^k is a quasi-contraction,

$$D^*(f^n x, u, u) \leq \frac{q^m}{1-q} D^*(f^j x, f^k f^j x, f^k f^j x)$$

$$\leq \frac{q^m}{1-q} \cdot \max\left\{ D^*(f^i x, f^k f^i x, f^k f^i x) : i = 0, 1, 2, \dots, k-1 \right\}$$

$$\leq \frac{q^m}{1-q} \cdot \max\left\{ D^*(f^i x, f^{k+i} x, f^{k+i} x) : i = 0, 1, 2, \dots, k-1 \right\}$$

proving (3.10). Letting $m \to \infty$, we get that $\lim_{n \to \infty} f^n x = u$, since $q^m \to 0$ as $m \to \infty$, proving (3.9). This completes the

proof of the theorem.

K. Theorem: Let f be a quasi-contraction with constant q on a metric space (X, d) and X be f-orbitally complete, then f has a unique fixed point $u \in X$. In fact,

$$u = \lim_{n \to \infty} f^n x$$
 for all $x \in X$

and

$$M. \ d\left(f^{n}x,u\right) \leq \frac{q^{n}}{1-q}.d\left(x,fx\right) \text{ for all } x \in X \ , \ n \geq 1.$$

1) *Proof:* If (X, d) is a *f*-orbitally complete metric space, then it can be proved that (X, D_1^*) is a *f*-orbitally complete *D**-metric space and hence *f*-orbitally complete for any selfmap *f* of *X*. Also if *f* is a quasi-contraction with constant *q* of (*X*, *d*), then the condition of quasi-contraction can be written as

$$D_{1}^{*}(fx, fy, fy) \leq q \cdot \max \left\{ D_{1}^{*}(x, y, y), D_{1}^{*}(x, fx, fx), D_{1}^{*}(y, fy, fy), D_{1}^{*}(x, fy, fy), D_{1}^{*}(y, fx, fx) \right\}$$



for all $x, y \in X$, since $D_1^*(x, y, y) = d(x, y)$; so that f is a quasi-contraction on (X, D_1^*) . Thus f is a quasi-contraction on the *f*-orbitally complete D^* -metric space (X, D_1^*) and hence the conclusions of Theorem 3.1 hold for f; which are the conclusions of the theorem.

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