iJRASET
International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 6 **Issue:** II **Month of publication:** February 2018

DOI:

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor : 6.887
Volume 6 Issue II, February 2018- Available at www.ijraset.com

Fixed Point Theorem and Consequences in D*-Metric Spaces

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Abstract: The purpose of this paper is to establish a fixed point theorem for quasi-contractions on D*-metric spaces and obtain certain consequences
Keywords: Quasi-contraction, generalized quasi-contraction, orbit of x under f of length n

I. INTRODUCTION

The purpose of this paper is to establish a fixed point theorem for quasi-contractions on *D**-metric spaces and obtain certain consequences. In fact, we prove that the fixed point theorem for quasi-contractions on metric spaces, proved by Lj. B. Ciric [1] as a particular case of the main result of this paper. The notion of Quasi-contraction defined for selfmaps of metric spaces given by Lj. B. Ciric [2] has been extended to the selfmaps of *D**-metric spaces. The notion of quasi-contractions has been extended to include a wider class of selfmaps of metric spaces by Fisher [3] and we distinguish them as generalized quasi-contractions. Here we define below generalized quasi-contractions among selfmaps of *D**-metric spaces (*X*, *D**).

II. PRELIMINARIES

A. Definition: A selfmap f of a *D**-metric space (*X*, *D**)is called a Quasi-contraction, if there is a number q with $0 \le q < 1$ such that

$$D^*(fx, fy, fy) \le q.\max\{ D^*(x,y,y), D^*(x,fx,fx), D^*(y,fy,fy), D^*(x,fy,fy), D^*(y,fx,fx) \}$$

for all $x, y \in X$.

B. Definition: A selfmap *f* of a *D**-metric space (*X*, *D**) is called a ***generalized quasi-contraction***, if for some fixed positive integers *k* and *l*, there is a number q with $0 \le q < 1$ such that

$$D^*(f^k x, f^l y, f^l y) \le q \max\{ D^*(f^r x, f^s y, f^s y), D^*(f^r x, f^{r'} x, f^{r'} x), D^*(f^s y, f^{s'} y, f^{s'} y) : 0 \le r, r' \le k; 0 \le s, s' \le l \}$$

for all $x, y \in X$.

C. Definition: Let f be a selfmap of a *D**-metric space (*X*, *D**) and $x \in X$, $n \ge 1$ be an integer. The orbit of x under f of length n, denoted by $O_f(x:n)$, is defined by

$$O_f(x:n) = \{x, fx, f^2x, \dots, f^n x\}$$

We define the diameter $\delta(A)$ of a set A in a *D**-metric space (*X*, *D**) by $\delta(A) = \sup_{x,y \in A}\{D^*(x,y,y)\}$

The following Lemmas are use full in proving fixed point theorems of quasi-contractions on *D**-metric spaces:

Suppose f is a quasi-contraction with constant q on a D^*-metric space (X, D^*) and n be a positive integer. Then for each $x \in X$ and all integers $i, j \in \{1, 2, 3, \ldots, n\}$,

$$D^*\left(f^i x, f^j x, f^j x\right) \le q.\delta\left[O_f(x:n)\right] < \delta\left[O_f(x:n)\right].$$

Let $x \in X$ be arbitrary, $n \ge 1$ be an integer and $i, j \in \{1, 2, 3, \ldots, n\}$. Then $f^{i-1}x, f^{j-1}x, f^i x, f^j x \in O_f(x:n)$ and since f is a quasi-contraction,

$$D^*\left(f^i x, f^j x, f^j x\right) = D^*\left(ff^{i-1}x, ff^{j-1}x, ff^{j-1}x\right)$$

$$\begin{aligned}
&\le q.\max\{D^*\left(f^{i-1}x, f^{j-1}x, f^{j-1}x\right), D^*\left(f^{i-1}x, f^i x, f^i x\right),\\
&\qquad D^*\left(f^{j-1}x, f^j x, f^j x\right), D^*\left(f^{i-1}x, f^j x, f^j x\right),\\
&\qquad D^*\left(f^{j-1}x, f^i x, f^i x\right)\}\\
&\le q.Sup\{D^*(u,v,v,): u,v \in O_f(x:n)\}\\
&= q.\delta\left[O_f(x:n)\right]\\
&< \delta\left[O_f(x:n)\right]
\end{aligned}$$

D. *Lemma:* Suppose f is a quasi-contraction with constant q on a D^*-metric space (X, D^*) and $x \in X$, then for every positive integer n, there exists positive integer $k \le n$, such that

$$D^*\left(x, f^k x, f^k x\right) = \delta\left[O_f(x:n)\right]$$

1) *Proof:* If possible assume that the result is not true. This implies that there is positive integer m such that for all $k \le m$, we have $D^*\left(x, f^k x, f^k x\right) \ne \delta\left[O_f(x:m)\right]$. Since $O_f(x:m)$ contains x and $f^k x$ for $k \le m$, it follows that

$$D^*\left(x, f^k x, f^k x\right) < \delta\left[O_f(x:m)\right]$$

Since $O_f(x:m)$ is closed, there exists $i, j \in \{1, 2, 3, \ldots, m\}$ such that $D^*\left(x, f^k x, f^k x\right) = \delta\left[O_f(x:m)\right]$, contradicting the Lemma 3.2.1. Therefore

$$D^*\left(x, f^k x, f^k x\right) = \delta\left[O_f(x:n)\right] \text{ for some } k \le n.$$

E. *Lemma:* Suppose f is a quasi-contraction with constant q on a D^*-metric space (X, D^*), then

$$\delta\left[O_f(x:\infty)\right] \le \frac{1}{1-q} D^*(x, fx, fx) \text{ for all } x \in X.$$

1) *Proof***:** Let $x \in X$ be arbitrary. Since $O_f(x:1) \subseteq O_f(x:2) \subseteq \ldots \subseteq O_f(x:n) \subseteq O_f(x:n+1) \subseteq \ldots$, we get that

$\delta\left[O_f(x:1)\right] \le \delta\left[O_f(x:2)\right] \le \ldots \le \delta\left[O_f(x:n)\right] \le \delta\left[O_f(x:n+1)\right] \le \ldots$, showing

$$\lim_{n\to\infty} \delta\left[O_f(x:n)\right] = Sup\left\{\delta\left[O_f(x:n)\right] : n = 1, 2, 3, \ldots\right\}.$$

III. MAIN RESULT

A. Theorem: Suppose f is a quasi-contraction with constant q on a D^*-metric space (X, D^*) and X is f-orbitally complete. Then f has a unique fixed point $u \in X$. In fact,

B. $u = \lim_{n\to\infty} f^n x$ for any $x \in X$

and

C. $D^*\left(f^n x, u, u\right) \le \dfrac{q^n}{1-q} D^*\left(x, fx, fx\right)$ for all $x \in X$, $n \ge 1$.

1) *Proof:* Let x be an arbitrary point of X. We claim that $\left\{f^n x\right\}$ is a Cauchy sequence in X. Let m, n be any positive integers with $n < m$. Since f is quasi-contraction,

$$D^*\left(f^n x, f^m x, f^m x\right) = D^*\left(ff^{n-1}x, ff^{m-1}x, ff^{m-1}x\right)$$

$$\le q.\delta\left[O_f\left(f^{n-1}x : m-n+1\right)\right]$$

That is,

D. $$D^*\left(f^n x, f^m x, f^m x\right) \le q.\delta\left[O_f\left(f^{n-1}x : m-n+1\right)\right]$$

According to the Lemma 2.3, there exists an integer k_1, with $0 \le k_1 \le m-n+1$, such that

E. $$\delta\left[O_f\left(f^{n-1}x : m-n+1\right)\right] = D^*\left(f^{n-1}x, f^{k_1} f^{n-1}x, f^{k_1} f^{n-1}x\right)$$

Using Lemma 2.4, we get

$$D^*\left(f^{n-1}x, f^{k_1} f^{n-1}x, f^{k_1} f^{n-1}x\right) = D^*\left(ff^{n-2}x, f^{k_1+1} f^{n-2}x, f^{k_1+1} f^{n-2}x\right)$$

$$\le q.\delta\left[O_f\left(f^{n-2}x : k_1+1\right)\right]$$

$$\le q.\delta\left[O_f\left(f^{n-2}x : m-n+2\right)\right]$$

(Since $k_1 + 1 \le m-n+2$)

Thus

F. $$D^*\left(f^{n-1}x, f^{k_1} f^{n-1}x, f^{k_1} f^{n-1}x\right) \le q.\delta\left[O_f\left(f^{n-2}x : m-n+2\right)\right]$$

From (3.4), (3.5) and (3.6) we get

$$D^*\left(f^{n}x, f^{m}x, f^{m}x\right) \le q.\delta\left[O_f\left(f^{n-1}x : m-n+1\right)\right]$$

$$\le q^2.\delta\left[O_f\left(f^{n-2}x : m-n+2\right)\right]$$

Proceeding in this manner, we obtain

$$D^*\left(f^{n}x, f^{m}x, f^{m}x\right) \le q^n.\delta\left[O_f\left(x : m\right)\right]$$

Using Lemma 2.3, we get

G. $$D^*\left(f^{n}x, f^{m}x, f^{m}x\right) \le \frac{q^n}{1-q}.\delta\left[O_f\left(x : m\right)\right]$$

Letting $n \to \infty$ and since $\lim_{n \to \infty} q^n = 0$, we get that $\left\{f^n x\right\}$ is Cauchy sequence. Again X being f-orbitally complete and $\left\{f^n x\right\}$ is a Cauchy sequence in $O_f\left(x : \infty\right)$, there is a point $u \in X$ such that $u = \lim_{n \to \infty} f^n x$.

We shall now show that u is a fixed point of f.

Consider,

$$D^*(u, fu, fu) \le D^*\left(u, f^{n+1}u, f^{n+1}u\right) + D^*\left(f^{n+1}u, fu, fu\right)$$

$$= D^*\left(u, f^{n+1}u, f^{n+1}u\right) + D^*\left(ff^{n}u, fu, fu\right)$$

$$\le D^*\left(u, f^{n+1}u, f^{n+1}u\right) + q.\max\{\; D^*\left(f^{n}u, u, u\right),$$

$$D^*\left(f^{n}u, f^{n+1}u, f^{n+1}u\right), D^*\left(u, fu, fu\right),$$

$$D^*\left(f^{n}u, fu, fu\right), D^*\left(u, f^{n+1}u, f^{n+1}u\right) \;\}$$

$$\leq D^*\left(u, f^{n+1}u, f^{n+1}u\right) + q.\left\{D^*\left(f^n u, f^{n+1}u, f^{n+1}u\right) + D^*\left(f^n u, u, u\right)\right.$$

$$\left. + D^*\left(u, fu, fu\right) + D^*\left(f^{n+1}u, u, u\right)\right\}$$

Letting $n \to \infty$ and since $\lim_{n\to\infty} f^n x = u$, we get

$D^*\left(u, fu, fu\right) = 0$ and hence $fu = u$, showing that u is fixed point of f.

To prove the uniqueness, let u, u' be two fixed points of f. That is, $fu = u,\ fu' = u'$

$$D^*\left(u, u', u'\right) = D^*\left(fu, fu', fu'\right)$$

$$\leq q.\max\left\{ D^*\left(u, u', u'\right), D^*\left(u, fu, fu\right), D^*\left(u', fu', fu'\right),\right.$$

$$\left. D^*\left(u, fu', fu'\right), D^*\left(u', fu, fu\right) \right\}$$

$$D^*\left(u, u', u'\right) \leq q.\max\left\{ D^*\left(u, u', u'\right), D^*\left(u, fu, fu\right), D^*\left(u', fu', fu'\right),\right.$$

$$\left. D^*\left(u, fu', fu'\right), D^*\left(u', fu, fu\right) \right\}$$

That is, $D^*\left(u, u', u'\right) \leq q.D^*\left(u, u', u'\right)$

Since $q < 1$, $D^*\left(u, u', u'\right) = 0$, which implies that $u = u'$.

Letting $n \to \infty$ in (3.7) we get (3.3). This completes the proof of the theorem.

H. *Theorem:* Suppose f is a selfmap of a D^*-metric space (X, D^*) and X is f-orbitally complete. If there is a positive integer k such that f^k is a quasi-contraction with constant q. Then f has a unique fixed point $u \in X$. In fact,

I. $u = \lim_{n\to\infty} f^n x$ for any $x \in X$

and

J. $D^*\left(f^n x, u, u\right) \leq \dfrac{q^n}{1-q} a(x)$ for all $x \in X$, $n \geq 1$,

where $a(x) = \max\left\{D^*\left(f^i x, f^{i+k} x, f^{i+k} x\right) : i = 1, 2, 3, \ldots\right\}$ and $m = \left[\dfrac{n}{k}\right]$, the greatest integer not exceeding $\dfrac{n}{k}$.

1) *Proof:* Suppose f^k is a quasi-contraction of a D^*-metric space (X, D^*). It has unique fixed point by Theorem 3.1. Let u be a fixed point of f^k. Then we claim that fu is also a fixed point of f^k. In fact,

$$f^k(fu) = f^{k+1}u = f\left(f^k u\right) = fu$$

By the uniqueness of fixed point of f^k, it follows that $fu = u$, showing that u is a fixed point of f. Uniqueness of the fixed point of f can be proved as in the Theorem 3.3.1.

To prove (3.10), let n be any integer. Then by the division algorithm, we have, $n = mk + j,\ 0 \le j < k,\ m \ge 0$

Therefore $x \in X,\ f^n x = \left(f^k\right)^m f^j x,$ since f^k is a quasi-contraction,

$$D^*\left(f^n x, u, u\right) \le \frac{q^m}{1-q} D^*\left(f^j x, f^k f^j x, f^k f^j x\right)$$

$$\le \frac{q^m}{1-q}.\max\left\{D^*\left(f^i x, f^k f^i x, f^k f^i x\right) : i = 0, 1, 2, \ldots, k-1\right\}$$

$$\le \frac{q^m}{1-q}.\max\left\{D^*\left(f^i x, f^{k+i} x, f^{k+i} x\right) : i = 0, 1, 2, \ldots, k-1\right\}$$

proving (3.10). Letting $m \to \infty$, we get that $\lim_{n \to \infty} f^n x = u$, since $q^m \to 0$ as $m \to \infty$, proving (3.9). This completes the proof of the theorem.

K. *Theorem:* Let f be a quasi-contraction with constant q on a metric space (X, d) and X be f-orbitally complete, then f has a unique fixed point $u \in X$. In fact,

L. $u = \lim_{n \to \infty} f^n x$ for all $x \in X$

and

M. $d\left(f^n x, u\right) \le \frac{q^n}{1-q}.d\left(x, fx\right)$ for all $x \in X$, $n \ge 1$.

1) *Proof:* If (X, d) is a f-orbitally complete metric space, then it can be proved that $\left(X, D_1^*\right)$ is a f-orbitally complete D^*-metric space and hence f-orbitally complete for any selfmap f of X. Also if f is a quasi-contraction with constant q of (X, d), then the condition of quasi-contraction can be written as

$$D_1^*\left(fx, fy, fy\right) \le q.\max\left\{D_1^*\left(x, y, y\right), D_1^*\left(x, fx, fx\right), D_1^*\left(y, fy, fy\right),\right.$$
$$\left. D_1^*\left(x, fy, fy\right), D_1^*\left(y, fx, fx\right)\right\}$$

for all $x, y \in X$, since $D_1^*(x, y, y) = d(x, y)$; so that f is a quasi-contraction on (X, D_1^*). Thus f is a quasi-contraction on the f-orbitally complete D^*-metric space (X, D_1^*) and hence the conclusions of Theorem 3.1 hold for f; which are the conclusions of the theorem.

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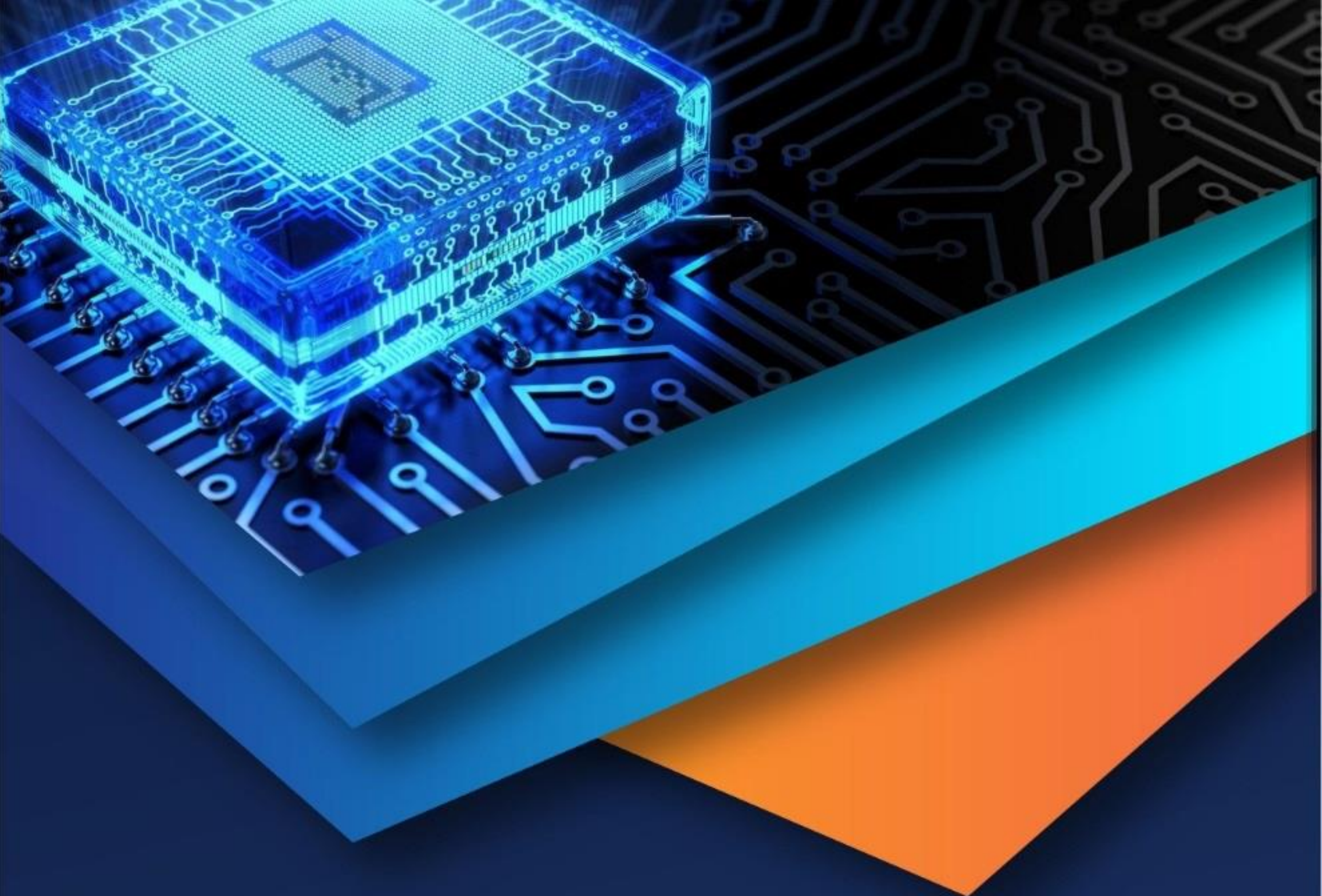

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