Fuzzy $\tau^*$-Generalized Closed Sets in Fuzzy Topological Spaces

M.K. Mishra$^1$, D. Anandhi$^2$, M. Prabhavathy$^3$

$^1$Director R&D, $^2,^3$Asst. Prof., E.G.S. Pillay Arts and Science College, Nagapattinam

Abstract: In this paper, we introduce a new class of sets called fuzzy $\tau^*$-generalized closed sets and fuzzy $\tau^*$-generalized open sets in fuzzy topological spaces and explore some of their properties.

Keywords: fuzzy closed set, fuzzy open set, fuzzy $\tau^*$-g-closed set, fuzzy $\tau^*$-g-open set.

I. INTRODUCTION

Let $X$ be a non-empty set and $I = [0,1]$. A fuzzy set on $X$ is a mapping from $X$ to $I$. The null fuzzy set is the mapping from $X$ to $I$ which always takes the values $0$ and whole fuzzy sets $I$ is a mapping from $X$ to $I$ which always takes the values $0$ and $1$. The union (resp. intersection) of a family $\{A_\alpha: \alpha \in \Lambda\}$ of fuzzy sets of $X$ is defined by to be the mapping $\sup A_\alpha$ (resp. inf $A_\alpha$). A fuzzy set $A$ of $X$ is contained in a fuzzy set $B$ of $X$ if $A(x) \leq B(x)$ for each $x \in X$. A fuzzy point $x_\beta$ in $X$ is a fuzzy set defined by $x_\beta(y) = \beta$ for $y = x$ and $x(y) = 0$ for $y \neq x$, $\beta \in [0,1]$ and $y \in X$. A fuzzy point $x_\beta$ is said to be quasi-coincident with the fuzzy set $A$ denoted by $x_\beta A$ if and only if $\beta + A(x) > 1$. A fuzzy set $A$ is quasi-coincident with a fuzzy set $B$ denoted by $A_\alpha B$ if and only if there exists a point $x \in X$ such that $A(x) + B(x) > 1$. A fuzzy set $A$ is a fuzzy set of $X$ if it is in the intersection of all fuzzy open subsets of $A$.

II. PRELIMINARIES

We recall the following definitions:

A. Definition 2.1: A subset $A$ of a fuzzy topological space $(X, \tau)$ is called:

1) Fuzzy Generalized closed (briefly fuzzy g-closed) if $cl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is fuzzy open in $X$.

2) Fuzzy Semi-generalized closed (briefly fuzzy sg-closed) if $scl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is fuzzy semiopen in $X$.

3) Fuzzy Generalized semi closed (briefly fuzzy gs-closed) if $scl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is fuzzy open in $X$.

4) Fuzzy-$\alpha$ closed (briefly fuzzy $\alpha$-closed) if $cl(int(cl(A))) \subseteq A$.

5) Fuzzy-$\alpha$-generalized closed (briefly fuzzy $\alpha$-g-closed) if $cl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is fuzzy open in $X$.

6) Fuzzy Generalized $\alpha$-closed (briefly fuzzy $\alpha$-g-closed) if $spcl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is fuzzy open in $X$.

7) Fuzzy Generalized semi-preclosed (briefly fuzzy gsp-closed) if $scl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is fuzzy open in $X$.

8) Fuzzy Strongly generalized closed (briefly fuzzy strongly g-closed) if $cl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is fuzzy open in $X$.

9) Fuzzy Preclosed if $cl(int(A)) \subseteq A$.

10) Fuzzy Semi-closed if $int(cl(A)) \subseteq A$.

11) Fuzzy Semi-preclosed (briefly fuzzy sp-closed) if $int(cl(int(A))) \subseteq A$.

The complements of the above mentioned sets are called their respective fuzzy open sets.

B. Definition 2.2: For the subset $A$ of a fuzzy topological $X$, the fuzzy generalized closure operator $cl^*[5]$ is defined by the intersection of all fuzzy $g$-closed sets containing $A$.

C. Definition 2.3: For the subset $A$ of a fuzzy topological $X$, the topology $\tau^*$ is defined by $\tau^* = \{G : cl^*(G) = G\}$.

D. Definition 2.4: For the fuzzy subset $A$ of a fuzzy topological $X$,

1) The fuzzy semi-closure of $A$ (briefly $scl(A)$) is defined as the intersection of all fuzzy semi-closed sets containing $A$.

2) The fuzzy semi-preclosure of $A$ (briefly $spcl(A)$) is defined as the intersection of all fuzzy semi-preclosed sets containing $A$.

3) The fuzzy $\alpha$-closure of $A$ (briefly $cl_\alpha(A)$) is defined as the intersection of all fuzzy $\alpha$-closed sets containing $A$.
III. Fuzzy $\tau^*$-Generalized Closed Sets in Fuzzy Topological Spaces

In this section, we introduce the concept of fuzzy $\tau^*$-generalized closed sets in fuzzy topological spaces.

A. Definition 3.1. A fuzzy subset $A$ of a fuzzy topological space $X$ is called fuzzy $\tau^*$-generalized closed set (briefly fuzzy $\tau^*$-closed) if $\text{cl}^*(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is fuzzy $\tau^*$-open. The complement of fuzzy $\tau^*$-generalized closed set is called the fuzzy $\tau^*$-generalized open set (briefly fuzzy $\tau^*$-open).

B. Theorem 3.1. Every fuzzy closed set in $X$ is fuzzy $\tau^*$-g-closed.

Proof: Let $A$ be a fuzzy closed set. Let $A \subseteq G$. Since $A$ is fuzzy closed, $\text{cl}(A) = A \subseteq G$. But $\text{cl}^*(A) \subseteq \text{cl}(A)$. Thus, we have $\text{cl}^*(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is fuzzy $\tau^*$-open. Therefore $A$ is fuzzy $\tau^*$-g-closed.

C. Theorem 3.2. Every fuzzy $\tau^*$-closed set in $X$ is fuzzy $\tau^*$-g-closed.

Proof: Let $A$ be a fuzzy $\tau^*$-closed set. Let $A \subseteq G$ where $G$ is fuzzy $\tau^*$-open. Since $A$ is fuzzy $\tau^*$-closed, $\text{cl}^*(A) = A \subseteq G$. Thus, we have $\text{cl}^*(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is fuzzy $\tau^*$-open. Therefore $A$ is fuzzy $\tau^*$-g-closed.

D. Theorem 3.3. Every fuzzy $g$-closed set in $X$ is a fuzzy $\tau^*$-g-closed set but not conversely.

Proof: Let $A$ be a fuzzy $g$-closed set. Assume that $A \subseteq G$, $G$ is fuzzy $\tau^*$-open in $X$. Then $\text{cl}(A) \subseteq G$, since $A$ is fuzzy $g$-closed. But $\text{cl}^*(A) \subseteq \text{cl}(A)$. Therefore $\text{cl}^*(A) \subseteq G$. Hence $A$ is fuzzy $\tau^*$-g-closed. The converse of the above theorem need not be true.

2) Remark 3.1.: The following example shows that fuzzy $\tau^*$-g-closed sets are independent from fuzzy sp-closed set, fuzzy sg-closed set, fuzzy $\alpha$-closed set, fuzzy precloser set fuzzy gs-closed set, fuzzy gsp-closed set, fuzzy ag-closed set and fuzzy ga-closed set.

E. Theorem 3.4.: For any two fuzzy sets $A$ and $B$, $\text{cl}^*(A \cup B) = \text{cl}^*(A) \cup \text{cl}^*(B)$

Proof: Since $A \subseteq A \cup B$, we have $\text{cl}^*(A) \subseteq \text{cl}^*(A \cup B)$ and since $B \subseteq A \cup B$, we have $\text{cl}^*(B) \subseteq \text{cl}^*(A \cup B)$. Therefore $\text{cl}^*(A) \cup \text{cl}^*(B) \subseteq \text{cl}^*(A \cup B)$. Also, $\text{cl}^*(A)$ and $\text{cl}^*(B)$ are the fuzzy closed sets. Therefore $\text{cl}^*(A) \cup \text{cl}^*(B)$ is also a fuzzy closed set.

Again, $A \subseteq \text{cl}^*(A)$ and $B \subseteq \text{cl}^*(B)$ implies $A \cup B \subseteq \text{cl}^*(A) \cup \text{cl}^*(B)$. Thus, $\text{cl}^*(A) \cup \text{cl}^*(B)$ is a closed set containing $A \cup B$. Since $\text{cl}^*(A \cup B)$ is the fuzzy smallest closed set containing $A \cup B$, we have $\text{cl}^*(A \cup B) \subseteq \text{cl}^*(A) \cup \text{cl}^*(B)$. Thus, $\text{cl}^*(A \cup B) = \text{cl}^*(A) \cup \text{cl}^*(B)$

F. Theorem 3.5.: Union of two fuzzy $\tau^*$ g-closed sets in $X$ is a fuzzy $\tau^*$-g-closed set in $X$.

Proof: Let $A$ and $B$ be two fuzzy $\tau^*$ g-closed sets. Let $A \cup B \subseteq G$, where $G$ is fuzzy $\tau^*$-open. Since $A$ and $B$ are fuzzy $\tau^*$-g-closed sets, $\text{cl}^*(A) \cup \text{cl}^*(B) \subseteq G$. But $\text{cl}^*(A) \cup \text{cl}^*(B) = \text{cl}^*(A \cup B)$. Therefore $\text{cl}^*(A \cup B) \subseteq G$. Hence $A \cup B$ is a fuzzy $\tau^*$-g-closed set.

G. Theorem 3.6.: A subset $A$ of $X$ is fuzzy $\tau^*$-g-closed if and only if $\text{cl}^*(A) = A$ contains non-empty fuzzy $\tau^*$-closed set in $X$.

Proof: Let $A$ be a fuzzy $\tau^*$-g-closed set. Suppose that $F$ is a nonempty fuzzy $\tau^*$-closed subset of $\text{cl}^*(A) = A$. Now $F \subseteq \text{cl}^*(A) = A$. Then $F \subseteq \text{cl}^*(A) \cap A^c$, since $\text{cl}^*(A) = A = \text{cl}^*(A) \cap A^c$. Therefore $F \subseteq \text{cl}^*(A)$ and $F \subseteq A^c$. Since $F$ is a fuzzy $\tau^*$-open set and $A$ is a fuzzy $\tau^*$-g-closed, $\text{cl}^*(A) \subseteq F$. That is $F \subseteq [\text{cl}^*(A)]^c$. Hence $F \subseteq \text{cl}^*(A) \cap [\text{cl}^*(A)]^c = \emptyset$. That is $F = \emptyset$, a contradiction. Thus $\text{cl}^*(A) = A$ contains no non-empty fuzzy $\tau^*$-closed set in $X$. Conversely, assume that $\text{cl}^*(A) = A$ contains no nonempty fuzzy $\tau^*$-closed set in $X$. Conversely, assume that $\text{cl}^*(A) = A$ contains no non-empty fuzzy $\tau^*$-closed set in $X$. Conversely, assume that $\text{cl}^*(A) = A$ contains no nonempty fuzzy $\tau^*$-closed set in $X$. Conversely, assume that $\text{cl}^*(A) = A$ contains no non-empty fuzzy $\tau^*$-closed set in $X$. Conversely, assume that $\text{cl}^*(A) = A$ contains no nonempty fuzzy $\tau^*$-closed set in $X$.

2) Corollary 3.1.: A subset $A$ of $X$ is fuzzy $\tau^*$-g-closed if and only if $\text{cl}^*(A) = A$ contains no non-empty fuzzy closed set in $X$.

Proof: Easy

4) Corollary 3.2.: A subset $A$ of $X$ is fuzzy $\tau^*$-g-closed if and only if $\text{cl}^*(A) = A$ contains no non-empty fuzzy open set in $X$.

5) Proof: The proof follows from the Theorem 3.10 and the fact that every open set is fuzzy $\tau^*$-open set in $X$.

H. Theorem 3.7.: If a subset $A$ of $X$ is fuzzy $\tau^*$-g-closed and $A \subseteq B \subseteq \text{cl}^*(A)$, then $B$ is fuzzy $\tau^*$-g-closed set in $X$. 
Proof: A be a fuzzy $\tau^*_g$-closed set such that $A \subseteq B \subseteq cl^*(A)$. Let $U$ be a fuzzy $\tau^*$-open set of $X$ such that $B \subseteq U$. Since $A$ is fuzzy $\tau^*_g$-closed, we have $cl^*(A) \subseteq U$. Now $cl^*(A) \subseteq cl^*(B) \subseteq cl^*[cl^*(A)] = cl^*(A) \subseteq U$. That is $cl^*(B) \subseteq U$, $U$ is fuzzy $\tau^*$-open. Therefore $B$ is fuzzy $\tau^*_g$-closed set in $X$. The converse of the above theorem need not be true.

REFERENCES

[21]. R. K. Saraf, M. Caldas and S. Mishra, Results via $F_g$-closed sets and $F_g$-closed sets, Pre print.