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# Independence of Mean Squares in Unbalanced Nested Designs 

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#### Abstract

In this paper, the distributional properties of independence of mean squares in three stage unbalanced nested designs with random effects have been investigated. The correlation coefficients between mean squares of main classes and mean squares of subclasses of different design structures have been calculated. Also the condones have been derived under which mean squares of main classes and mean squares of sub classes are independently distributed Keywords: unbalanced, nested, mean squares, independence, correlations, designs


## I. INTRODUCTION AND LITERATURE SURVEY

In balanced three stage nested classifications with random effects, the mean squares in the analysis of variance are independent and are chi-squares distributed under the usual normality assumptions. The variance component estimators are linear functions of mean squares and their variances can be derived accordingly. When the classification data are not balanced, the mean squares, in general , are no longer independent and also they do not follow chi-square type distributions. Similarly the distributions of variance component estimators of classes and subclasses are not linear combinations of chi-squares in unbalanced classification. An exact test does not always exist even in balanced designs for random effects model. The ANOVA testing procedures were extended by [1], [2] and [3] for the construction of approximate F-tests in unbalanced designs. These tests make use of the fact that the mean squares are distributed independently and have chi-square type distributions but that do not generally hold for unbalanced designs. Ignoring all these factors, onle appromate tests developed for balanced designs have also been applied to unbalanced designs. [4] have discussed approximate F-tests for testing the variance components in which correlation of mean squares of numerator and synthesized denominator of F- test are calculated. Further, they have calculated correlations of mean squares in some special cases. There is a need to study the dependence of mean squares on all types of imbalance in nested designs. Further, [6] has mentioned that zero covariance of mean squares always implies independence under the normality assumptions. Therefore, an empirical study of twenty designs of different structures is undertaken and calculated the correlations and derived the conditions under which the mean squares are independently distributed. Consider the three stage unbalanced nested random effects model :

$$
\mathrm{i}=1,2, \ldots \ldots \mathrm{a}, \quad \mathrm{j}=1,2, \ldots \ldots \ldots . . b_{i}
$$

$$
\begin{gathered}
y_{i j k}=\mu+a_{i}+b_{i j}+e_{i j k} \\
k=1,2, \ldots \ldots \ldots \ldots \ldots n_{i j}
\end{gathered}
$$

Where $\mathrm{y}_{\mathrm{ijk}}$ is the observation, $\mu$ is the general mean, $\mathrm{a}_{\mathrm{i}}$ and $\mathrm{b}_{\mathrm{ij}}$ are the random effects, $\mathrm{e}_{\mathrm{ijk}}$ the random error associated with $\mathrm{y}_{\mathrm{ijk}}$. As usual, it is assumed that $a_{i} \sim N\left(0, \sigma_{a}^{2}\right), b_{i j} \sim N\left(0, \sigma_{b}^{2}\right), e_{i j k} \sim N\left(0, \sigma_{e}^{2}\right)$

The ANOVA for three stage unbalanced nested design with random effects is given in Table 1 below :
Table 1
Analysis of Variance (ANOVA)
Source of variation d.f Mean Squares Expected Mean Squares

| Main classes | $\mathrm{f}_{3}$ | $\mathrm{y}^{\prime} \mathrm{Q}_{3} \mathrm{y} / \mathrm{f}_{3}=\mathrm{V}_{3}$ | $\sigma_{e}^{2}+q_{1} \sigma_{b}^{2}+q_{2} \sigma_{a}^{2}$ |
| :--- | :--- | :--- | :--- |
| Sub classes | $\mathrm{f}_{2}$ | $\mathrm{y}^{\prime} \mathrm{Q}_{2} \mathrm{y} / \mathrm{f}_{2}=\mathrm{V}_{2}$ | $\sigma_{e}^{2}+q_{0} \sigma_{b}^{2}$ |

Within subclassesf $_{1}$

$$
y^{\prime} \mathrm{Q}_{1} \mathrm{y} / \mathrm{f}_{1}=\mathrm{V}_{1}
$$

$$
\sigma_{e}^{2}
$$

Where
$\mathrm{Q}_{3}=\sum_{i}^{+} \frac{J_{n_{i}}}{n_{i}}-J_{n} / n$
$\mathrm{Q}_{2}=\sum_{i} \sum_{j}^{+} J_{n_{i j}} / n_{i j} \quad-\sum_{i}^{+} \frac{J_{i}}{n_{i}}$,
$\mathrm{Q}_{1}=\mathrm{I}-\sum_{i} \sum_{j}^{+} J_{n_{i j}} / n_{i j}$
$\mathrm{q}_{0}=\sum_{i} \sum_{j} n_{i j}^{2}\left(1 / n_{i j}-1 / n_{i}\right) / \mathrm{f}_{2} \quad, \quad \mathrm{q}_{1}=\sum_{i} \sum_{j} n_{i j}^{2}\left(1 / n_{i}-1 / n\right) / \mathrm{f}_{3}$,

$$
\mathrm{q}_{2}=\sum_{i} n_{i}^{2}\left(1 / n_{i}-1 / n\right) / \mathrm{f}_{3}
$$

also $\quad \sum_{i} b_{i}=b$,
$\sum_{j} n_{i j}=n_{i}$,
$\sum_{i} \sum_{j} n_{i j}=\sum_{i} n_{i}=n$
$\mathrm{f}_{1}=\mathrm{n}-\sum_{i} b_{i} \quad \mathrm{f}_{2}=\sum_{i} b_{i}-a \quad \mathrm{f}_{3}=\mathrm{a}-1$

And $\mathrm{J}_{\mathrm{n}}$ is the $n \times n$ matrix of 1 's and $\sum_{i=1}^{k+} A_{i}$ is the direct sum of matrices $\mathrm{A}_{1}, \mathrm{~A}_{2} \ldots \ldots \ldots . . \mathrm{A}_{k}$ as defined in [6].
The designs are given below in Table 1.
Table 1

|  |  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Design1 | $\mathrm{b}_{\mathrm{i}}$ | 2 | 2 | 2 | 2 |  |  |  |  | $\mathrm{a}=4$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 2,1 | 2,1 | 2,1 | 2,1 |  |  |  |  |  |
| Design2 | $\mathrm{b}_{\mathrm{i}}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | $a=8$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 2,1 | 2,1 | 2,1 | 2,1 | 2,1 | 2,1 | 2,1 | 2,1 |  |
| Design3 | $\mathrm{b}_{\mathrm{i}}$ | 2 | 2 | 2 | 2 |  |  |  |  | $\mathrm{a}=4$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 20,1 | 20,1 | 20,1 | 20,1 |  |  |  |  |  |
| Design4 | $\mathrm{b}_{\mathrm{i}}$ | 2 | 3 | 2 | 1 |  |  |  |  | $\mathrm{a}=4$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 2,2 | 2,2,2 | 2,2 | 2 |  |  |  |  |  |
| Design5 | $\mathrm{b}_{\mathrm{i}}$ | 2 | 3 | 2 | 1 |  |  |  |  | $\mathrm{a}=4$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 3,3 | 3,3,3 | 3,3 | 3 |  |  |  |  |  |
| Design6 | $\mathrm{b}_{\mathrm{i}}$ | 5 | 4 | 2 | 5 |  |  |  |  | $a=4$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 2,2,2,2,2 | 2,2,2,2 | 2,2 | 2,2,2,2,2,2,2 |  |  |  |  |  |
| Design7 | $\mathrm{b}_{\mathrm{i}}$ | 15 | 2 | 1 |  |  |  |  |  | $\mathrm{a}=3$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | $\begin{aligned} & \hline 2,2,2,2,2,2,2,2, \\ & 2,2,2,2,2,2,2 \\ & \hline \end{aligned}$ | 2,2 | 2 |  |  |  |  |  |  |
| Design8 | $\mathrm{b}_{\mathrm{i}}$ | 2 | 1 | 1 | 1 | 1 |  |  |  | $a=5$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 4,4 | 2 | 2 | 2 | 2 |  |  |  |  |
| Design9 | $\mathrm{b}_{\mathrm{i}}$ | 1 | 2 | 3 | 4 |  |  |  |  | $a=4$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 4 | 3,3 | 2,2,2 | 1,1,1,1 |  |  |  |  |  |
| Design10 | $\mathrm{b}_{\mathrm{i}}$ | 3 | 4 | 2 | 3 | 2 |  |  |  | $\mathrm{a}=5$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 4,4,4 | 2,2,2,2 | 1,1 | 3,3,3 | 5,5 |  |  |  |  |
| Design11 | $\mathrm{b}_{\mathrm{i}}$ | 2 | 2 | 2 | 2 | 1 | 1 |  |  | $\mathrm{a}=6$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 2,2 | 1,1 | 1,1 | 1,1 | 1 | 1 |  |  |  |
| Design12 | $\mathrm{b}_{\mathrm{i}}$ | 2 | 2 | 2 | 2 | 1 | 1 |  |  | $a=6$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 1,1 | 1,1 | 1,1 | 1,1 | 2 | 2 |  |  |  |


| Design13 | $\mathrm{b}_{\mathrm{i}}$ | 2 | 2 | 1 | 1 | 1 | 1 |  |  | $\mathrm{a}=6$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 1,1 | 1,1 | 2 | 2 | 2 | 2 |  |  |  |
| Design14 | $\mathrm{b}_{\mathrm{i}}$ | 1 | 1 | 3 |  |  |  |  |  | $\mathrm{a}=3$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 1 | 2 | $4,4,4$ |  |  |  |  |  |  |
| Design15 | $\mathrm{b}_{\mathrm{i}}$ | 1 | 1 | 3 |  |  |  |  |  | $\mathrm{a}=3$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 5 | 4 | $2,2,2$ |  |  |  |  |  |  |
| Design16 | $\mathrm{b}_{\mathrm{i}}$ | 1 | 1 | 3 |  |  |  |  |  |  |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 1 | 5 | $3,3,3$ | $\mathrm{a}=3$ |  |  |  |  |  |
| Design17 | $\mathrm{b}_{\mathrm{i}}$ | 1 | 2 | 2 |  |  |  |  |  |  |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 7 | 3,3 | 1,1 |  |  |  |  |  | $\mathrm{a}=3$ |
| Design18 | $\mathrm{b}_{\mathrm{i}}$ | 2 | 1 | 1 | 1 | 1 | 1 |  |  | $\mathrm{a}=6$ |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 8,8 | 2 | 2 | 2 | 2 | 2 |  |  |  |
| Design19 | $\mathrm{b}_{\mathrm{i}}$ | 4 | 4 | 4 |  |  |  |  |  |  |
|  | $\mathrm{n}_{\mathrm{ij}}$ | $1,8,12,2$ | $7,1,15,3$ | $10,2,4,6$ |  |  |  |  |  |  |
| Design20 | $\mathrm{b}_{\mathrm{i}}$ | 2 | 4 | 6 | 1 |  |  |  |  |  |
|  | $\mathrm{n}_{\mathrm{ij}}$ | 1,5 | $2,7,1,3$ | $1,8,12,3,20,2$ | 5 |  |  |  |  |  |

Where ' $a$ ' indicates the number of main classes, $b_{i}$ indicates the number of subclasses within the main class and $n_{i j}$ the number of observations.

## II. RESULTS AND DISCUSSION

Correlations between $V_{2}$ and $V_{3}$ in three stage unbalanced nested designs are given in Table 2 :
Table 2

|  | Design 1 |  |  |  |  |  |  |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :---: |
| X | Y |  |  |  |  |  |  |
|  | 0 | 0.5 | 1 | 4 | 16 | 24 |  |
|  | 0 | 0.0157 | 0.0309 | 0.0634 | 0.0707 | 0.0819 |  |
|  | 0.5 | 0 | 0.0087 | 0.0198 | 0.0530 | 0.0756 |  |
|  |  |  |  |  |  |  |  |
|  | 0 | 0.0060 | 0.0146 | 0.0456 | 0.0719 | 0.0763 |  |
|  | 0 | 0.0021 | 0.0056 | 0.0247 | 0.0556 | 0.0634 |  |
| 16 | 0 | 0.0006 | 0.0016 | 0.0087 | 0.0292 | 0.0377 |  |
|  | 0 | 0.0004 | 0.0011 | 0.0061 | 0.0221 | 0.0297 |  |


| X | Design 2 |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\begin{array}{llll} & & & \mathrm{Y} \\ 0 & 0.5 & 1 & 4\end{array}$ |  |  |  | 16 | 24 |
|  |  |  |  |  |  |  |
|  | 0 | 0.0170 | 0.0334 | 0.0685 | 0.0861 | 0.0885 |
| 0 | 0 | 0.0094 | 0.0214 | 0.0573 | 0.0817 | 0.0854 |
| 0.5 | 0 | 0.0065 | 0.0157 | 0.0492 | 0.0777 | 0.0825 |
| 1 | 0 | 0.0023 | 0.0061 | 0.0267 | 0.0601 | 0.0685 |
| 4 | 0 | 0.0006 | 0.0018 | 0.0094 | 0.0315 | 0.0408 |
| 16 | 0 | 0.0004 | 0.0012 | 0.0064 | 0.0239 | 0.0321 |


|  | Design 3 |  |  |  |  |  |
| ---: | :---: | :---: | :---: | :--- | :--- | :--- |
| X | Y |  |  |  |  |  |
|  | 0 | 0.5 | 1 | 4 | 16 | 24 |
|  | 0 | 0.1721 | 0.2429 | 0.3401 | 0.3762 | 0.3806 |
|  | 0 | 0.0863 | 0.1595 | 0.2995 | 0.3631 | 0.3721 |
|  | 0 | 0.0575 | 0.1188 | 0.2675 | 0.3521 | 0.3640 |
|  | 0 | 0.0192 | 0.0469 | 0.1631 | 0.2953 | 0.3218 |
|  | 0 | 0.0052 | 0.0137 | 0.0637 | 0.1795 | 0.2198 |
|  | 0 | 0.0035 | 0.0093 | 0.0453 | 0.1423 | 0.1815 |


| X | Designs 4 to 18 |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Y |  |  |  |  |  |
|  | 0 | 0.5 |  |  |  |  |
| $\downarrow$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 |
| $16$ | 0 | 0 | 0 | 0 | 0 | 0 |


| X | Design 19 |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Y |  |  |  |  |  |
|  | 0 | 0.5 |  | 1 | 4 | 16 |
| $\downarrow$ | 24 |  |  |  |  |  |
| 0 | 0 | 0.0826 | 0.1028 | 0.1229 | 0.1287 | 0.1294 |
| 0.5 | 0 | 0.0264 | 0.0474 | 0.0934 | 0.1191 | 0.1228 |
| 1 | 0 | 0.0157 | 0.0308 | 0.0753 | 0.1109 | 0.1168 |
| 4 | 0 | 0.0046 | 0.0099 | 0.0349 | 0.0783 | 0.0905 |
| 16 | 0 | 0.0012 | 0.0027 | 0.0111 | 0.0360 | 0.0476 |
| 24 | 0 | 0.0008 | 0.0018 | 0.0076 | 0.0265 | 0.0361 |


|  | Design 20 |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | Y |  |  |  |
| X | 0 | 0.5 | 1 | 4 | 16 | 24 |
| $\downarrow$ | 0 | 0.0619 | 0.0784 | 0.0951 | 0.1001 | 0.1006 |
| 0 | 0 | 0.0223 | 0.0391 | 0.0745 | 0.0934 | 0.0961 |
| 0.5 | 0 | 0.0135 | 0.0258 | 0.0610 | 0.0876 | 0.0919 |
| 1 | 0 | 0.0040 | 0.0085 | 0.0290 | 0.0634 | 0.0724 |
| 4 | 0 | 0.0010 | 0.0023 | 0.0093 | 0.0298 | 0.0392 |
| 16 | 0 | 0.0007 | 0.0015 | 0.0064 | 0.0220 | 0.0299 |

In this study, the ratio of $2 / 2$ and $2 / 2$ are denoted by X and Y respectively. The designs 1 and 2 ( Table 1 ) are of special kind in which $b_{i}=2, n_{i 1}=2$ and $n_{i 2}=1$ for all $i$. In these designs, correlation between mean squares $V_{2}$ and $V_{3}$ increases for given $X$
and decreases as $X$ increases for given $Y$. The magnitude of correlation is very small for $X \geq 4$ and $\mathrm{Y} \leq 4$ (Table 2 ). When we increase the number of classes in design 2 , the corresponding correlations are slightly increased. When we increase the imbalance at the third stage as in design 3 i.e $n_{i 1}=20$ and $n_{i 2}=1$ with $b_{i}=2$ for all $i$, the correlation increases 4 to 10 times. Further, it is noticed that for each given value of X , the values of correlation decreased as the value of Y increased. A similar trend is also noticed for each given value of $Y$, the values of correlation decreased as $X$ increased in design 3. When $n_{i j}=k_{i}$ for all $i$, that is, balance within the classes, the correlation between mean squares $V_{2}$ and $V_{3}$ are all zero for all the values of $X$ and $Y$. This shows that mean squares $V_{2}$ and $V_{3}$ are independent distributed variables In literature, such designs are called partially balanced nested designs. The designs 4 to 7 are of this kind .When $\mathrm{n}_{\mathrm{ij}}=k$ for all i and j , that is, balance at the last stage ( $\mathrm{b}_{\mathrm{i}}$ 's are different) , the correlation between mean squares $\mathrm{V}_{2}$ and $V_{3}$ are all zero for all the values of $X$ and $Y$. Again, this shows that mean squares $V_{2}$ and $V_{3}$ are independent distributed variables in designs 8 to 18 . Such designs are called last stage uniformity designs. The design 19 is balanced at the second stage but highly unbalanced at the third stage. The design 20 is general unbalanced one and the correlations are given in Table 2.

## III. CONCLUSION

We have noticed that correlation between $V_{2}$ and $V_{3}$ are zero when $Y={ }^{2} /{ }^{2}=0$ for $X \geq 0$ in all the twenty designs. This shows that under the acceptance of the hypothesis that ${ }^{2}$ is equal to zero, the mean squares $V_{2}$ and $V_{3}$ are independently distributed. Further, When $n_{i j}=k$ for all $i$ and $j$, that is, balance at the last stage ( $b_{i}$ 's are different) and $n_{i j}=k_{i}$ for all $i$, that is, balance within the classes, the mean squares $\mathrm{V}_{2}$ and $\mathrm{V}_{3}$ are independently distributed variables.

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