iJRASET
International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH
IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 6 **Issue:** II **Month of publication:** February 2018

DOI:

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor : 6.887
Volume 6 Issue II, February 2018- Available at www.ijraset.com

A Comparative Study on Numerical Solutions of [IVP] for[ODE] with Cubic Spline Interpolation and Modified Euler Method

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Abstract: This paper mainly presents,Cubic Spline interpolation and Modified Euler method for solving initial value problems [IVP] for ordinary differential equations [ODE]. The two proposed methods are quite efficient and practically well suited for solving these problems. In order to verify the accuracy, we compare numerical solutions. Numerical comparison between Cubic spline interpolation and Modified Euler method have been presented. Also we compare the Natural Cubic spline interpolation and clamped spline interpolation and computational effort of such methods. In order to achieve higher accuracy in the solution, the step size needs to be very small. Finally we investigate and compute the errors of the two proposed methods for different, with based on the intervals to examine superiority
Keywords: Initial value problem [IVP], Natural Cubic Spline interpolation, Clamped Cubic Spline interpolation, Modified Euler method, Error analysis.

I. INTRODUCTION

Differential equations are commonly used for mathematical modeling in science and engineering. Many problems of mathematical physics can be started in the form of differential equations. These equations also occur as reformulations of other mathematical problems such as ordinary differential equations and partial differential equations. In most real life situations, the differential equation that models the problem is too complicated to solve exactly, and one of two approaches is taken to approximate the solution. The first approach is to simplify the differential equation to one that can be solved exactly and then use the solution of the simplified equation to approximate the solution to the original equation. The other approach, which we will examine in this paper, uses methods for approximating the solution of original problem. This is the approach that is most commonly taken since the approximation methods give more accurate results and realistic error information.

II. PROBLEM FORMULATION

In this section we consider two numerical methods for finding the approximate solutions of the initial value problem (IVP) of the first-order ordinary differential equation has the form

$$y'' = (x, y(x)),$$
$$y(x_0) = y_0$$

In this paper we determine the solution of this equation on a finite interval (x_0, x_n) starting with the initial point x_0.A continuous approximation to the solution *y(x)* will not be obtained; instead, approximations to *y* will be generated at various values, called mesh points, in the interval(x_0, x_n). Numerical methods employ the Equations to obtain approximations to the values of the solution corresponding to various selected values of

$$X = X_n = X_0 + nh, n = 1,2,3, \dots$$

The parameter *h* is called the step size. The numerical solutions is given by a set of points $\{(x_n, y_n)$: n= 0,1, 2, …,n$\}n$ and each point$(x_n, y_n)$is an approximation to the correspondingpoint$(x_n, y(x_n))$on the solution curve.

III. CUBIC SPLINE INTERPOLATION:

Given nodes and data f(x_0, f (x_0)), (x_1, f (x_1)),…, (x_n, f(x_n))we have interpolated using

A. Lagrange interpolating polynomial of degree n, with n + 1 coefficients,

B. Such polynomials can possess large oscillations, and the error term can be difficult to construct and estimate.

C. A cubic polynomial $s(x) = a + bx + cx^2 + dx^3$ is specified by 4 coefficients.
D. The cubic spline is twice continuously differentiable.
E. The cubic spline has the flexibility to satisfy general types of boundary conditions.
F. While the spline may agree with f (x) at the nodes, we cannot guarantee the derivatives of the spline agree with the derivatives of f .

Given a function f (x) defined on [a, b] and a set of nodes

$$a = x_0 < x_1 < x_3 < \ldots < x_n = b;$$

a cubic spline interpolate S, for f is a piecewise cubic polynomial, S_j on $[\, x_j, x_{j+1}]$ for j = 0, 1,…, n – 1

$$s(x) = \begin{cases} a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 & if x_0 \le x \le x_1 \\ a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 & if x_1 \le x \le x_2 \\ . \\ . \\ . \\ a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 & if x_{n-1} \le x \le x_n \end{cases}$$

The cubic spline interpolate will have the following properties.

$(a) S_j(x_i) = f(x_j) and S_j(x_{j+1}) = f(x_{j+1}) for each j = 0, 1, \ldots, n - 1;$

$(b) S_{j+1}(x_{j+1}) = S_j(x_{j+1}) for each j = 0, 1, \ldots, n - 2$

$(c) S'_{j+1}(x_{j+1}) = S'_j(x_{j+1}) for each j = 0, 1, \ldots, n - 2$

$(d) S''_{j+1}(x_{j+1}) = S''_j(x_{j+1}) for each j = 0, 1, \ldots, n - 2$

(f) One of the following boundary conditions (BCs) is satisfied:

$(i) S''(x_0) = S''(x_n) = 0$ (free or natural BCs).

$$(ii) S'(x_0) = f'(x_0) and S'(x_n) = f'(x_n) \quad (Clamped boundary)$$

Although Cubic splines are defined with other boundary conditions, the conditions given in (f) are sufficient for our purposes. When the free boundary conditions occur, the spline is called a Natural spline, and its graph approximates the shape that a long flexible rod would assume if forced to go through the data points.

IV. MODIFIED EULER METHOD

Modified Euler method is the simplest one-step method. It is basic explicit method for numerical integration of ordinary differential equations. Euler proposed his method for initial value problems (IVP) in 1768. It is first numerical method for solving IVP and serves to illustrate the concepts involved in the advanced methods. It is important to study because the error analysis is easier to understand. The general formula for Modified Euler approximation

$$y_{n+1} = y_n + h\left(f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hf(x_n, y_n)\right)\right), \quad n = 1,2,...$$

V. ERROR ANALYSIS

here are two types of errors in numerical solution of ordinary differential equations. Round-off errors and Truncation errors occur when ordinary differential equations are solved numerically. Rounding errors originate from the fact that computers can only represent numbers using a fixed and limited number of significant figures. Thus, such numbers or cannot be represented exactly in computer memory. The discrepancy introduced by this limitation is call Round-off error. Truncation errors in numerical analysis arise when approximations are used to estimate some quantity. The accuracy of the solution will depend on how small we make the step size *h*. A numerical method is said to be convergent if

$$\lim_{h \to 0} \max_{1 \le n \le N} |y(x_n) - y_n| = 0.$$

Where $y(x_n)$ denotes the approximate solution and y_n denotes exact solution.

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
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VI. NUMERICAL EXAMPLES

In this section we consider two numerical examples to prove which numerical methods converge faster to analytical solution. Numerical results and errors a are computed and the outcomes are represented by graphically.

1.Use the data points (0,1), (1,e), (2,e^2) and (3,e^3) to form a Cubic Spline $S(x)$that approximate $f(x)$=e^x.

Solution: (Natural cubic spline) We have n=3, h_0=h_1=h_2=1

x	$f(x)$
0	1
1	e
2	e^2
3	e^3

And we take a_0=1, a_1=e, a_2=e^2, a_3=e^3So the matrix A and the vector b and x given in below,

$$A=\begin{bmatrix}1&0&0&0\\1&4&1&0\\0&1&4&1\\0&0&0&1\end{bmatrix} b=\begin{bmatrix}0\\3(e^2-2e+1)\\3(e^3-2e^2+e)\\0\end{bmatrix} \quad \text{and} \quad x=\begin{bmatrix}c_0\\c_1\\c_2\\c_3\end{bmatrix}$$

$$\boldsymbol{b_j=\frac{a_{j+1}-a_j}{h_j}-h_j(c_{j+1}+2c_j)/3}$$

$$\boldsymbol{d_j=(c_{j+1}-c_j)/3h_j}$$

j	a_j	b_j	c_j	d_j
0	1	1.465998113	0	0.252284214
1	2.718281828	2.372851258	0.756851144	1.691071369
2	7.389056099	8.809769923	5.830066751	-1.943355584

Therefore Natural cubic spline is described piecewise by,

( ) = + (–) + (–) + (–)

∫ **=19.47728687**

A. *Clamped Cubic Spline*

0=1, 1= , 2= 2, 3= 3 We have n=3, h_0=h_1=h_2=1, by the given information the matrix form is,

$$A=\begin{bmatrix}2&1&0&0\\1&4&1&0\\0&1&4&1\\0&0&1&2\end{bmatrix}; \quad b=\begin{bmatrix}3(\ \ -2)\\3(\ ^2-2\ \ +1\\3(\ ^3-2\ ^2+\ \ )\\3\ ^2\end{bmatrix} \quad \text{and} \quad x=\begin{bmatrix}_0\\_1\\_2\\_3\end{bmatrix}$$

0	1	1.592945094	0.44470882024	0.273573007
1	2.718281828	2.710136664	1.26542784351	0.695209763
2	7.389056099	7.3242403	3.35105713471	2.018999489

Therefore Clamped cubic spline is described piecewise by

( ) = + (−) + (−) + (−)

∫ **=19.35610106**

Numerical approximations and maximum errors for based on the piecewise approximation. A numerical method is said to be convergent if

$\leq \leq$ | ( ) − | =

where ( ) denotes the approximate solution and denotes exact solution.

Piecewise interval	Exact solution	Natural cubic spline	Error	Clamped cubic spline	Error
[0,1]	1.7182800	1.796070109	0.07790109	1.998207063	0.075927063
[1,2]	4.6707741	4.579759030	0.09101509	4.664743789	0.006030311
[2,3]	12.696481	12.10445585	0.59202497	12.56233516	0.134145662

Now the approximation value for Natural spline interpolation in the interval [0,3] is to be 19.47728687
And the approximation value for Clamped spline interpolation in the interval [0,3] is to be 19.35610106

B. Modified Euler Method

Numerical approximation for the Modified Euler method for step size h=1

0	1
1	2.648721271
2	7.130410341
3	19.31290432

∫ = .

Exact solution	Clamped cubic spline	Error	Modified Euler	Error
19.08553692	19.3561016	0.27056468	19.3129043	0.22736739

From the comparison of two numerical methods Modified Euler method found to be less error.

C. Use the data points (0,1), (3,2), (8,3) to form a Cubic Spline S( )that approximate ( )=√ + 1.

Natural cubic spline) We have n=2, $h_0=3, h_1=5$

	( )
0	*1*
3	*2*
8	*3*

And we take $_0=1$, $_1=2$, $_2=3$, So the matrix A and the vector b and x given in below,

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A=$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 16 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ b=$\begin{bmatrix} 0 \\ 0.4 \\ 0 \end{bmatrix}$ and x=$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$

0	1	0.3583	0	0.00278
1	2	0.28333	- 0.025	0.00167

Therefore Natural cubic spline is described piecewise by,

$(\ ) = \ + \ (\ - \ ) + \ (\ - \ ) + \ (\ - \ )$

$\int \sqrt{ \ + \ }$ =**17.3178495**

n the same manner of the Natural cubic spline,

$_0$=1, $_1$=2, $_2$=3 $'(0) = 0.5$, $'(8) = 0.166667$

We have n=2, h_0=3, h_2=8, by the given information the matrix form is,

A=$\begin{bmatrix} 6 & 3 & 0 \\ 3 & 16 & 5 \\ 0 & 5 & 10 \end{bmatrix}$; b=$\begin{bmatrix} -0.5 \\ -0.4 \\ -0.1 \end{bmatrix}$ and x=$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

0	1	0.499996733	-0.0791667	0.007870745
1	2	0.237498333	- 0.00833	0.0001667

Therefore Clamped cubic spline is described piecewise by,

$(\ ) = \ + \ (\ - \ ) + \ (\ - \ ) + \ (\ - \ )$

$\int \sqrt{ \ + \ }$ =**17.3045602**

Numerical approximations and maximum errors for based on the piecewise approximation.

$\underset{\to \quad \le \ \le}{} | \ (\ ) - \ | =$

Piecewise interval	Exact solution	Natural cubic spline	Error	Clamped cubic spline	Error
[0,1]	0.9428090	1.7182326	0.2354236	1.1258410	0.1830320
[1,2]	1.5784835	1.5309541	0.0475294	1.5709115	0.0075720
[2,3]	1.8692317	3.2563759	1.3871442	3.1259512	1.2567194
[3,4]	2.1202266	2.1338325	0.0236059	2.2164170	0.0141510
[4,5]	2.3443990	2.27284075	0.0715583	2.3370330	0.0073659
[5,6]	2.5488048	4.61096175	2.0620813	4.4601176	1-9112371
[6,7]	2.7381052	5.02922575	2.2911205	5.0228312	2.2847260
[7,8]	2.9150573	7.63166925	4.7166119	7.5748608	4.6598035

Now the approximation value for Natural spline interpolation in the interval [0,8] is to be 17.3178495
And te approximation value for Clamped spline interpolation in the interval [0,8] is to b e17.3045602

D. Modified Euler Method

Numerical approximation for the Modified Euler method for step size h=1

0	1
1	2.2247450
2	3.8058800
3	5.6767120
4	7.7980329
5	10.143241
6	12.692751
7	15.431363
8	18.346839

$\int \sqrt{\quad + \quad}$ **=18.346839**

Exact solution	Clamped cubic spline	Error	Modified Euler	Error
18.0000000	17.3045602	0.69543979	18.3468391	0.3468391

.
From the comparison of two numerical methods Modified Euler method found to be less error. Finally we observe that the Modified Euler method is converging faster than the clamped spline interpolation and it is most effective method for solving initial value problems for ordinary differential equations.

VII. CONCLUSION

In this paper, Cubic spline interpolation and Modified Euler method are used for solving ordinary differential equation (ODE)in initial value problems (IVP). Finding more accurate results needs the step size smaller for all methods. From the figures we can see the accuracy of the methods for *h*and the graph of the approximate solution approaches to the graph of the exact solution. The numerical solutions obtained by the two proposed methods are in good agreement with exact solutions. Comparing the results of the two methods under investigation, we observed that the rate of convergence of Clamped cubic spline interpolation and the rate of convergence of Modified Euler method. The Clamped cubic spline interpolation was found to be less accurate due to the in accurate numerical results that were obtained from the approximate solution in comparison to the exact solution

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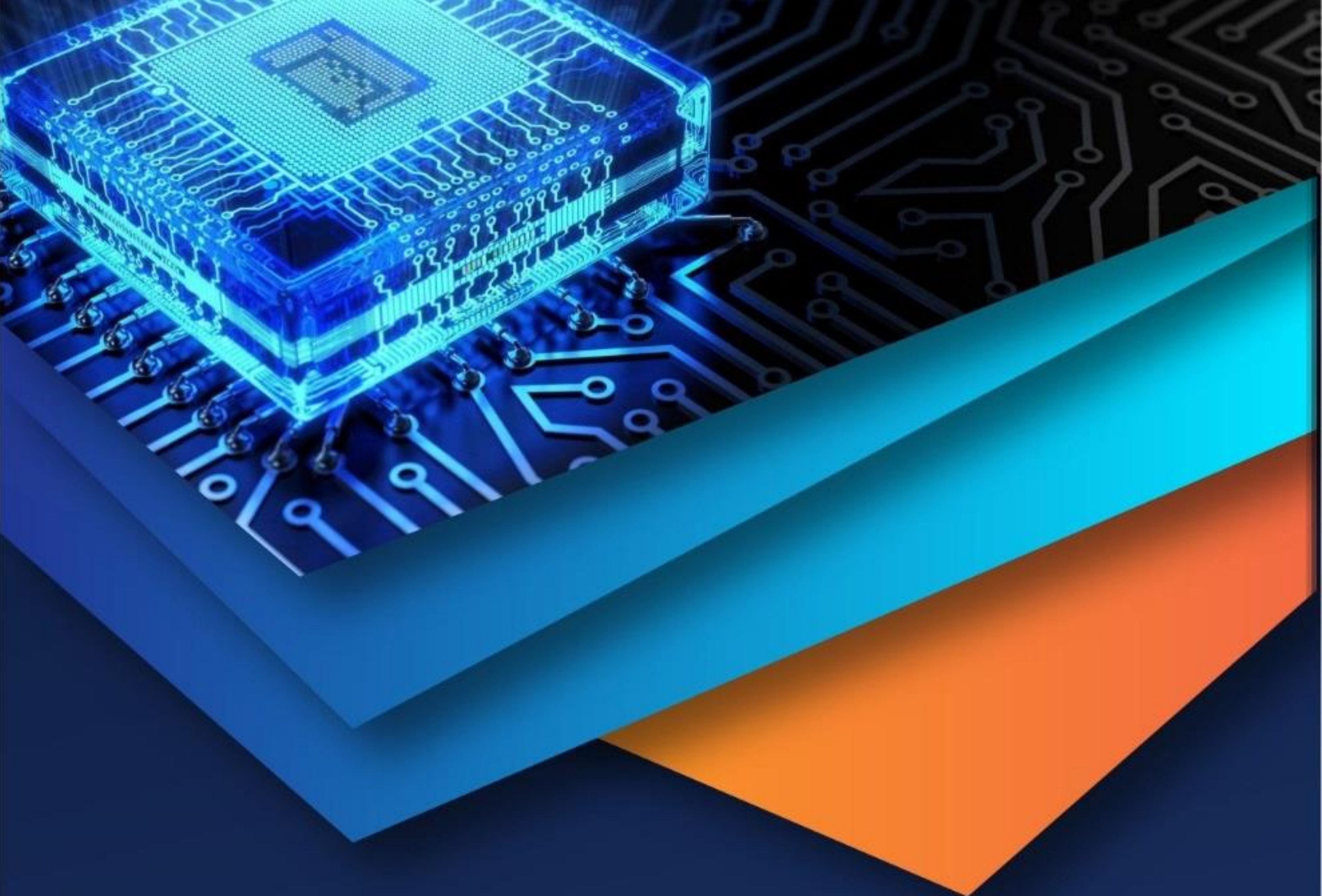

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