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Generalized Fixed Point Theorem for Quasi-Contractions

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Abstract: ***In this paper we proved a fixed point theorem for generalized quasi-contractions of generalized metric spaces.***
***Keywords: D-metric space* or a *generalized metric space, Quasi-contraction, Generalized quasi-contraction*.**

I. INTRODUCTION

In this paper a fixed point theorem for generalized quasi-contractions of *D**-metric spaces have been proved. The notion of quasi-contractions on *D**-metric spaces has been generalized by Brian Fisher [1]. Analogously we define generalized quasi-contractions among the selfmaps of *D**-metric spaces. Generally fixed point theorems were established for self maps of metric spaces. Certain fixed point theorems were proved for self maps of metrizable topological spaces also since such spaces, for all practical purposes, can be considered as metric spaces. B. C. Dhage [2] has initiated a study of general metric spaces called *D*-metric spaces. Later several researchers have made a significant contribution to the fixed point theorems of D-metric spaces. As a probable modification of D-metric spaces, very recently, ShabanSedghi, NabiShobe and Haiyun Zhou [3] have introduced D*-metric spaces.

II. PRELIMINARIES

*A. Definition 2.1:*Let Xbe a non-empty set. A function $D^*: X^3 \to [0,\infty)$ is said to be a generalized metric or D*-metricon X, if it satisfies the following conditions:

$D^*(x,y,z) \geq 0$ for all $x,y,z \in X$

$D^*(x,y,z) = 0$ if and only if $x = y = z$

$D^*(x,y,z) = D^*(\sigma(x,y,z))$ for all $x,y,z \in X$,

$\sigma(x,y,z)$ is a permutation of the set $\{x,y,z\}$

$D^*(x,y,z) \leq D^*(x,y,w) + D^*(w,z,z)$ for all $x,y,z,w \in X$.

The pair (*X*, *D**), where *D** is a generalized metric on *X* is called a *D**-metric space or a generalized metric space.

B. Definition2.2: A selfmap*f* of a *D**-metric space (*X*, *D**) is called aquasi-contraction, if there is a number q with $0 \leq q < 1$ such that

$$\begin{aligned} D^*(fx,fy,fy) \leq q\ \max\{ & D^*(x,y,y), D^*(x,fx,fx), D^*(y,fy,fy), \\ & D^*(x,fy,fy), D^*(y,fx,fx) \} \end{aligned}$$

for all $x,y \in X$.

C. Definition2.3: A self map*f* of a *D**-metric space (*X*, *D**) is called a generalized quasi-contraction, if for some fixed positive integers k and l, there is a number q with $0 \leq q < 1$ such that

$$\begin{aligned} D^*\left(f^k x, f^l y, f^l y\right) \leq q.\max\{ & D^*\left(f^r x, f^s y, f^s y\right), D^*\left(f^r x, f^{r'} x, f^{r'} x\right), \\ & D^*\left(f^s y, f^{s'} y, f^{s'} y\right) : 0 \leq r, r' \leq k,\ 0 \leq s, s' \leq l \ \} \end{aligned}$$

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for all $x, y \in X$.

III. MAIN RESULT

A. *Theorem:* Suppose *f* is a generalized quasi-contraction on a complete *D**-metric spaces (*X* , *D**) and let f be continuous. Then f has a unique fixed point.

1) *Proof:* By increasing the value of q if necessary, we may assume that $\frac{1}{2} \le q < 1$ and the inequality in the generalized quasi contraction will still hold. But we will then have that $\frac{q}{1-q} \ge 1$. Assume that $k \ge l$

Let *x* be an arbitrary point in *X*, we claim that the sequence $\left\{ f^n x : n = 1, 2, 3, \ldots \right\}$ is bounded.

If possible assume that the sequence $\left\{ f^n x : n = 1, 2, 3, \ldots \right\}$ is unbounded. Then the sequence $\left\{ D^*\left(f^n x, f^l x, f^l x \right) : n = 1, 2, 3, \ldots \right\}$ is unbounded.

Let $K = \frac{q}{1-q}.\max\left\{ D^*\left(f^i x, f^l x, f^l x \right) : 0 \le i \le k \right\}$. Since $\left\{ D^*\left(f^n x, f^l x, f^l x \right) : n = 1, 2, 3, \ldots \right\}$ is unbounded, there exists an integer *n* such that $D^*\left(f^n x, f^l x, f^l x \right) > K$ and let n_0 be smallest such n and since $\frac{q}{1-q} \ge 1$, $n_0 > k \ge l$. Thus

$$D^*\left(f^{n_0} x, f^l x, f^l x \right) > K = \frac{q}{1-q}.\max\left\{ D^*\left(f^i x, f^l x, f^l x \right) : 0 \le i \le k \right\}$$

That is,

B.
$$D^*\left(f^{n_0} x, f^l x, f^l x \right) > \frac{q}{1-q}.\max\left\{ D^*\left(f^i x, f^l x, f^l x \right) : 0 \le i \le k \right\}$$

Now it follows that

$$\begin{aligned}
(1-q) D^*\left(f^{n_0} x, f^l x, f^l x \right) &> q.\max\left\{ D^*\left(f^i x, f^l x, f^l x \right) : 0 \le i \le k \right\} \\
&\ge q.\max\left\{ D^*\left(f^i x, f^r x, f^r x \right) - D^*\left(f^r x, f^l x, f^l x \right) : \right. \\
&\qquad \left. 0 \le i \le k;\ 0 \le r < n_0 \right\} \\
&\ge q.\max\left\{ D^*\left(f^i x, f^r x, f^r x \right) - D^*\left(f^{n_0} x, f^l x, f^l x \right) : \right. \\
&\qquad \left. 0 \le i \le k;\ 0 \le r < n_0 \right\} \\
&= q.\max\left\{ D^*\left(f^i x, f^r x, f^r x \right) : 0 \le i \le k;\ 0 \le r < n_0 \right\} \\
&\qquad - q.D^*\left(f^{n_0} x, f^l x, f^l x \right)
\end{aligned}$$

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And hence

C. $$D^*\left(f^{n_0}x, f^lx, f^lx\right) > q.\max\left\{ D^*\left(f^ix, f^rx, f^rx\right) : 0 \le i \le k;\ 0 \le r < n_0 \right\}$$

We shall now prove that

D. $$D^*\left(f^{n_0}x, f^lx, f^lx\right) > q.\max\left\{ D^*\left(f^ix, f^rx, f^rx\right) : 0 \le i, r < n_0 \right\}$$

If not,

E. $$D^*\left(f^{n_0}x, f^lx, f^lx\right) \le q.\max\left\{ D^*\left(f^ix, f^rx, f^rx\right) : k < i, r < n_0 \right\}$$

Now applying the inequality in the definition of the generalized quasi-contraction repeatedly to the inequality (3.5), we get

$$D^*\left(f^ix, f^rx, f^rx\right) = D^*\left(f^k f^{i-k}x, f^l f^{r-l}x, f^l f^{r-l}x\right)$$

$$\le q.\max\Big\{ D^*\left(f^{r_1+i-k}x, f^{s_1+r-l}x, f^{s_1+r-l}x\right),$$

$$D^*\left(f^{r_1+i-k}x, f^{r_1'+i-k}x, f^{r_1'+i-k}x\right),$$

$$D^*\left(f^{s_1+r-l}x, f^{s_1'+r-l}x, f^{s_1'+r-l}x\right):$$

$$0 \le r_1, r_1' \le k; 0 \le s_1, s_1' \le l \Big\}$$

$$= q.\max\Big\{ D^*\left(f^{r_1+i-k}x, f^{s_1+r-l}x, f^{s_1+r-l}x\right),$$

$$D^*\left(f^{r_1+i-k}x, f^{r_1'+i-k}x, f^{r_1'+i-k}x\right),$$

$$D^*\left(f^{s_1+r-l}x, f^{s_1'+r-l}x, f^{s_1'+r-l}x\right):$$

$$0 \le r_1 + i - k \le n_0, k - l \le s_1 + r - l \le n_0,$$
$$0 \le r_1' + i - k \le n_0, k - l \le s_1' + r - l \le n_0 \Big\}$$

$$= q.\max\left\{D^*\left(f^px, f^tx, f^tx\right), D^*\left(f^px, f^{p'}x, f^{p'}x\right),\right.$$

$$\left. D^*\left(f^tx, f^{t'}x, f^{t'}x\right) : k \le p, p', t, t' \le n_0\right\},$$

Omitting the terms of the form $D^*\left(f^i x, f^r x, f^r x\right)$ with $0 \le i \le k$, because of inequality (3.3).

Thus

$$\begin{aligned} D^*\left(f^{n_0} x, f^l x, f^l x\right) &\le q.\max\left\{ D^*\left(f^i x, f^r x, f^r x\right) : k < i, r < n_0 \right\} \\ &\le q^2.\max\left\{ D^*\left(f^i x, f^r x, f^r x\right) : k < i, r < n_0 \right\} \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ &\le q^m.\max\left\{ D^*\left(f^i x, f^r x, f^r x\right) : k < i, r < n_0 \right\} \end{aligned}$$

for $m = 1, 2, 3, \ldots$ and on letting $m \to \infty$, it follows that

$D^*\left(f^{n_0} x, f^l x, f^l x\right) = 0$, which is a contradiction. Therefore the inequality (3.4) holds. However, we now have

$$D^*\left(f^{n_0} x, f^l x, f^l x\right) = D^*\left(f^k f^{n_0 - k} x, f^l x, f^l x\right)$$

and on using inequality in the definition of the generalized quasi-contraction, we have

$$\begin{aligned} D^*\left(f^{n_0} x, f^l x, f^l x\right) \le q.\max\Big\{ & D^*\left(f^{r+n_0-k} x, f^s x, f^s x\right), \\ & D^*\left(f^{r+n_0-k} x, f^{r'+n_0-k} x, f^{r'+n_0-k} x\right), \\ & D^*\left(f^s x, f^{s'} x, f^{s'} x\right) : 0 \le r, r' \le k; 0 \le s, s' \le l \Big\} \\ = q.\max\Big\{ & D^*\left(f^p x, f^s x, f^s x\right), \\ & D^*\left(f^p x, f^{p'} x, f^{p'} x\right), \\ & D^*\left(f^s x, f^{s'} x, f^{s'} x\right) : n_0 - k \le p, p' \le n_0; 0 \le s, s' \le l \Big\} \end{aligned}$$

(since, $0 \le r \le k$ implies $n_0 - k \le r + n_0 - k \le n_0$)

Therefore

$$\begin{aligned} D^*\left(f^{n_0} x, f^l x, f^l x\right) \le q.\max\Big\{ & D^*\left(f^p x, f^s x, f^s x\right), \\ & D^*\left(f^p x, f^{p'} x, f^{p'} x\right), D^*\left(f^s x, f^{s'} x, f^{s'} x\right) : \\ & n_0 - k \le p, p' \le n_0; 0 \le s, s' \le l \Big\} \end{aligned}$$

$$\leq q.\max\left\{ D^*\left(f^r x, f^s x, f^s x\right) : 0 \leq r, s \leq n_0 \right\}$$

And this is impossible because of the inequality (3.4). Hence we get that the sequence $\left\{ f^n x : n = 1, 2, 3, \ldots \right\}$ is bounded for any $x \in X$. This implies that $\left\{ D^*\left(f^r x, f^s x, f^s x\right) : r \geq 0, s \geq 0 \right\}$ is bounded. Therefore $M = Sup\left\{ D^*\left(f^r x, f^s x, f^s x\right) : r, s = 0, 1, 2, 3, \ldots \right\} < \infty$. Since, $q < 1$, $q^n \to 0$ as $n \to \infty$, so that for every $\varepsilon > 0$, there is a natural number N such that $q^n < \dfrac{\varepsilon}{M}$ for all $n \geq N$. In particular, $q^N.M < \varepsilon$. Let $N_0 = N.\max\{k, l\}$. If $m \geq N_0$, $n \geq N_0$, $D^*\left(f^m x, f^n x, f^n x\right) < q^{N_0}.M < q^N.M < \varepsilon$.

Thus $\left\{ f^n x : n = 1, 2, 3, \ldots \right\}$ is a Cauchy sequence in the complete *D**-metric space (*X*, *D**) and hence has a limit, say, *u*. That is, $u = \lim_{n \to \infty} f^n x$. As f is continuous, this implies that $fu = f\left(\lim_{n \to \infty} f^n x\right) = \lim_{n \to \infty} f^{n+1} x = u$, and so u is a fixed point of f.

To prove the uniqueness of u, let $u' \in X$ be such that $fu' = u'$. From inequality in the generalized quasi-contraction, we get

$$D^*\left(u, u', u'\right) = D^*\left(f^k u, f^l u', f^l u'\right)$$

$$\leq q.\max\left\{ D^*\left(f^r u, f^s u', f^s u'\right),\right.$$

$$D^*\left(f^r u, f^{r'} u, f^{r'} u\right), D^*\left(f^s u', f^{s'} u', f^{s'} u'\right) :$$

$$\left. 0 \leq r, r' \leq k; 0 \leq s, s' \leq l \right\}$$

$$= q.D^*\left(u, u', u'\right)$$

and since $0 \leq q < 1$, we get that $D^*\left(u, u', u'\right) = 0$ which implies that $u = u'$. Thus the fixed point of f is unique in *X*.

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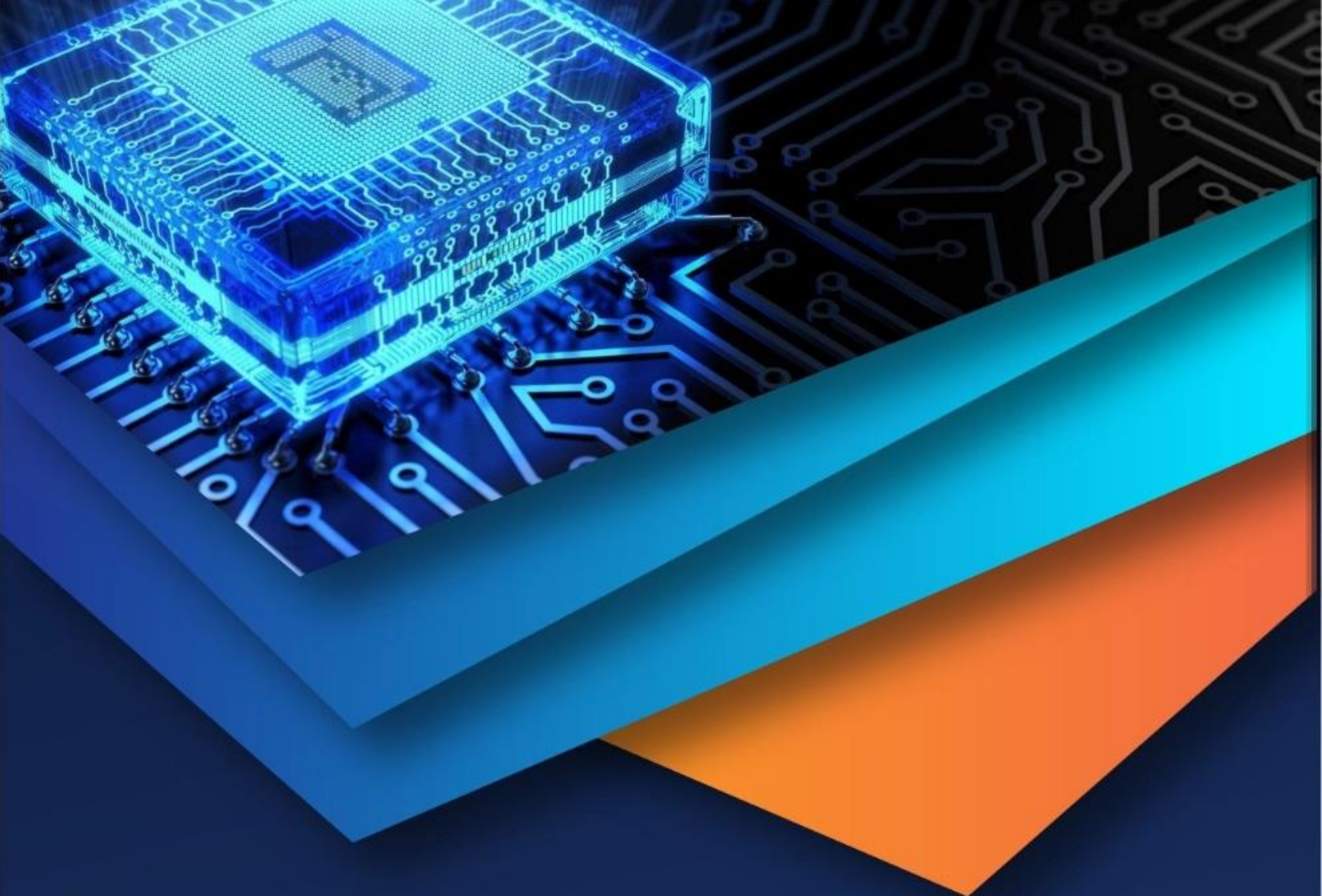

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